An Introduction to Numerical Analysis of SPDEs

by István Gyöngy

Maxwell Institute and School of Mathematics
University of Edinburgh

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Motivations, Nonlinear Filtering

$$\text{State: } dX_t = b(Z_t)\,dt + \theta(Z_t)\,dW_t + \rho(Z_t)\,dV_t, \quad X_0 = \xi$$

$$\text{Observation: } dY_t = B(Z_t)\,dt + dV_t, \quad Y_0 = \eta,$$

$Z_t := (X_t, Y_t) \in \mathbb{R}^{d+d_1}$, (W, V) is a multi-dim. Wiener process;
b, B are Lipschitz continuous vector fields,
θ, ρ are Lipschitz continuous matrix fields on $\mathbb{R}^{d+d_1}$;
(ξ, η) is a random vector, independent of (W, V).

Task: Calculate the mean square estimate $\hat{\phi}$ of $\varphi(X_t)$ for a bounded function φ, given $(Y_s)_{s\in[0,t]} =: Y_{[0,t]}$, i.e.,

$$E|\hat{\varphi} - \varphi(X_t)|^2 = \min_f E|f(Y_{[0,t]}) - \varphi(X_t)|^2 .$$

Clearly,

$$\hat{\varphi} = P_t(\varphi) := E(\varphi(X_t)|Y_s, s \leq t) = \int_{\mathbb{R}^d} \varphi(x) P_t(dx)$$

$$= \int_{\mathbb{R}^d} \varphi(x) \pi_t(x)\, dx,$$

where

$$P_t(dx) := P(X_t \in dx | Y_s, s \leq t) = \pi_t(x)\, dx.$$

Questions:

1. Equations for P_t, π_t?
2. Existence and regularity of π_t?
3. How to calculate π_t numerically?
4. Robustness of the filter?

1. Equations of nonlinear filtering

Notation:

$$b_t(x) := b(x, Y_t), \quad \rho_t(x) = \rho(x, Y_t), \quad \theta_t(x) := \theta(x, Y_t),$$
$$B_t(x) = B(x, Y_t), \quad a_t(x) := (\rho_t \rho_t^*(x) + \theta_t \theta_t^*(x))/2, \quad x \in \mathbb{R}^d$$

$$L_t = a_t^{ij}(x) D_i D_j + b_t^i(x) D_i, \quad M_t^r = \rho_t^{ir}(x) D_i + B_t^r(x)$$
$$L_t^* \psi = D_i D_j (a_t^{ij} \psi) - D_i (b_t^i \psi), \quad M_t^{r*} \psi = -D_i (\rho_t^{ir} \psi) + B_t^{ir} \psi.$$

Filtering equation (*Kushner-Shiryayev equation*):

$$dP_t(\varphi) = P_t(L\varphi)\, dt + \{P_t(M^r \varphi) - P_t(B_t^r) P_t(\varphi)\}\, d\bar{V}_t^r$$
$$P_0(\varphi) = E(\varphi(X_0)),$$

$d\bar{V}_t = dY_t - P_t(B_t)\, dt$, the "innovation process".

If $\pi_t(x) := P_t(dx)/dx$ exists, then it satisfies

$$d\pi_t(x) = L_t^* \pi_t(x)\, dt + \{M_t^{r*} \pi_t(x) - (\pi_t, B_t^r) \pi_t(x)\}\, d\bar{V}_t^r,$$
$$\pi_0(x) = P(X_0 \in dx)/dx =: p_0,$$

Transform the *Kushner-Shiryayev equation* by $Q_t(\varphi) := \lambda_t P_t(\varphi)$,

$$\lambda_t := \exp\left(- \int_0^t P_s(B_s)\, d\bar{V}_s - \tfrac{1}{2} \int_0^t |P_s(B_s)|^2\, ds \right)$$

$$= \exp\left(- \int_0^t P_s(B_s)\, dY_s + \tfrac{1}{2} \int_0^t |P_s(B_s)|^2\, ds \right).$$

Then $d\lambda_t = \lambda_t P_t(B_t^k)\, dY_t^k$, and by Itô's formula

$$dQ_t(\varphi) = P_t(\varphi)\, d\lambda_t + \lambda_t\, dP_t(\varphi) + dP_t(\varphi)\, d\lambda_t$$

$$= Q_t(\varphi) P_t(B_t)\, dY_t + Q_t(L_t\varphi)\, dt + (Q_t(M_t\varphi) - Q_t(\varphi)P_t(B_t))\, d\bar{V}_t$$
$$+ (Q_t(M_t\varphi) - Q_t(\varphi)P_t(B_t))\, P_t(B_t) dt$$

$$= Q_t(L_t\varphi)\, dt + Q_t(M_t\varphi)\, dY_t.$$

Thus for $Q_t = \lambda_t P_t$ we get a linear equation, called *Zakai equation*,

$$dQ_t(\varphi) = Q_t(L_t\varphi)\,dt + Q_t(M_t^r\varphi)\,dY_t^r, \quad Q_0(\varphi) = P_0(\varphi).$$

Clearly, $Q_t(1) = P_t(1)\lambda_t = \lambda_t$. Hence $P_t(\varphi) = Q_t(\varphi)/Q_t(1)$.
If $u_t(x) = Q_t(dx)/dx$, the unnormalised density exists, then

$$d(u_t, \varphi) = (u_t, L_t\varphi)\,dt + (u_t, M_t^k\varphi)\,dY_t^k, \quad (u_0, \varphi) = (p_0, \varphi).$$

By integration by parts, we have

$$d(u_t, \varphi) = (L_t^* u_t, \varphi)\,dt + (M_t^{k*} u_t, \varphi)\,dY_t^k$$

for all $\varphi \in C_0(\mathbb{R}^d)$, i.e., $u = u_t(x)$ solves

$$\begin{aligned} du_t(x) =& L_t^* u_t(x)\,dt + M_t^{k*} u_t(x)\,dY_t^k, \\ u_0(x) =& p_0(x). \end{aligned}$$

The Zakai equation can be obtained also as follows:
Define $(\gamma_t)_{t\in[0,T]}$ and $\bar{P}$ by

$$d\gamma_t = -\gamma_t B^k(Z_t)\, dV_t^k, \quad \gamma_0 = 1, \quad d\bar{P} = \gamma_T\, dP.$$

Then $(Y_t - Y_0, W_t)_{t\in[0,T]}$ is an $\mathcal{F}_t$-Wiener martingale under $\bar{P}$, and

$$P_t(\varphi) = E(\varphi(X_t)|\mathcal{Y}_t) = \frac{\bar{E}(\varphi(X_t)\gamma_T^{-1}|\mathcal{Y}_t)}{\bar{E}(\gamma_T^{-1}|\mathcal{Y}_t)} = \frac{\mu_t(\varphi)}{\mu_t(1)}, \quad t \in [0, T],$$

where $\mu_t(\varphi) := \bar{E}(\varphi(X_t)\gamma_T^{-1}|\mathcal{Y}_t)$.
Note that

$$d\gamma_t^{-1} = \gamma_t^{-1} B^k(Z_t)\, dY_t^k.$$

Hence $(\gamma_t^{-1})_{t\in[0,T]}$ is an $\mathcal{F}_t$-martingale under $\bar{P}$. Thus

$$\mu_t(\varphi) = \bar{E}(\varphi(X_t)\gamma_t^{-1}|\mathcal{Y}_t), \quad t \in [0, T].$$

By Itô's formula

$$d(\varphi(X_t)\gamma_t^{-1}) = \varphi(X_t)\, d\gamma_t^{-1} + \gamma_t^{-1}\, d\varphi(X_t) + d\varphi(X_t)\, d\gamma_t^{-1}.$$

$$= \gamma_t^{-1} B^r(Z_t)\varphi(X_t)\, dY_t^r$$

$$+\gamma_t^{-1} L_t\varphi(X_t)\, dt + \gamma_t^{-1}\theta^{ik}(Z_t) D_i\varphi(X_t)\, dW_t^k + \gamma_t^{-1}\rho^{ir}(Z_t) D_i\varphi(X_t)\, dV_t^r$$

$$+\gamma_t^{-1}\rho^{ir}(Z_t) D_i\varphi(X_t) B^r(Z_t)\, dt$$

$$= \gamma_t^{-1} L_t\varphi(X_t)\, dt + \gamma_t^{-1}\theta^{ik}(Z_t) D_i\varphi(X_t)\, dW_t^k + \gamma_t^{-1} M_t^r\varphi(X_t)\, dY_t^r.$$

Thus

$$d(\varphi(X_t)\gamma_t^{-1}) = \gamma_t^{-1} L_t\varphi(X_t)\,dt + \gamma_t^{-1}\theta^{ik}(Z_t)D_i\varphi(X_t)\,dw_t^k$$
$$+\gamma_t^{-1} M_t^r\varphi(X_t)\,dY_t^r.$$

Taking conditional expectation $\bar{E}(\cdot|\mathcal{Y}_t)$, we get

$$d\mu_t(\varphi) = \mu_t(L_t\varphi)\,dt + \mu_t(M_t^r\varphi)\,dY_t^r,$$

$$\mu_0(\varphi) = \bar{E}(\varphi(X_0)|\mathcal{Y}_0) = E(\gamma_T\varphi(X_0)|\mathcal{Y}_0) = E(\varphi(X_0)|\mathcal{Y}_0) = P_0(\varphi).$$

2. Existence of the conditional density

- Existence of a solution $u = u_t(x)$ to the Zakai equation

$$du_t(x) = L_t^* u_t(x)\, dt + M_t^{*k} u_t(x)\, dY_t^k, \quad u_0(x) = P(X_0 \in dx|\mathcal{Y})/dx$$

- Uniqueness of measure-valued solutions to

$$\mu_t(\varphi) = \mu_t(L_t\varphi)\, dt + \mu_t(M_t^k\varphi), dY_t^k, \quad P_0 = E(\varphi(X_0)|\mathcal{Y})$$

in a class of finite measures containing P_t, $P_t(dx) = u_t(x)\, dx$.
Hence u_t is the unnormalised density, and

$$\pi_t(x) = \frac{u_t(x)}{\int_{\mathbb{R}^d} u_t\, dx}$$

is the conditional density of X_t given $\{Y_s, s \le t\}$.

Remark. There is a direct approach, developed by Krylov, to the existence of the conditional density and to investigate its properties: One can directly show that

$$E(\varphi(X_t)|\mathcal{Y}_t) = \bar{\gamma}_t\,(u_t, \varphi), \quad \varphi \in C_0^\infty,$$

where

$$d\gamma_t = -\gamma_t B^k(Z_t)\,dV_t^k, \quad \gamma_0 = 1, \quad \bar{\gamma}_t := E(\gamma_t|\mathcal{Y}_t),$$

and u is the solution to the Zakai equation.

Recall

$$\begin{aligned} dX_t &= b(X_t, Y_t)\, dt + \theta(X_t, Y_t)\, dW_t + \rho(X_t, Y_t)\, dV_t, \\ dY_t &= B(X_t, Y_t)\, dt + \, dV_t \end{aligned}$$

$$b_t(x) := b(x, Y_t), \quad \rho_t(x) = \rho(x, Y_t), \quad \theta_t(x) := \theta(x, Y_t),$$

$$B_t(x) = B(x, Y_t), \quad a_t(x) := (\rho_t \rho_t^*(x) + \theta_t \theta_t^*(x))/2,$$

$$L_t^* \psi = D_i D_j (a_t^{ij} \psi) - D_i (b_t^i \psi) = D_i (a_t^{ij} D_j \psi) + D_i (\bar{b}_t^i \psi), \quad \bar{b}_t^i = D_j a^{ij} - b^i,$$

$$M_t^{r*} \psi = -D_i (\rho_t^{ir} \psi) + B_t^r \psi = -\rho^{ir} D_i \psi + \bar{B}_t^r \psi, \quad \bar{B}^i = B^r - D_i B^r.$$

The Zakai equation is a second order parabolic (possibly degenerate) SPDE: For $z = (z^1, z^2, ..., z^d) \in \mathbb{R}^d$

$$\tilde{a}^{ij} z^i z^j := (a^{ij} - \frac{1}{2} \rho^{ir} \rho^{jr}) z^i z^j = \frac{1}{2} \theta^{ik} \theta^{jk} z^i z^j = \sum_j |\theta^{ik} z^j|^2 \geq 0.$$

On solvability of second order parabolic SPDEs

Consider

$$
\begin{aligned}
du_t(x) =& (\mathcal{L}_t u_t(x) + f_t(x))\, dt + (\mathcal{M}^r_t u_t(x) + g^r_t(x))\, dW^r_t, && (1)\\
u_0(x) =& \psi(x) && (2)
\end{aligned}
$$

for $t \in [0, T]$, $x \in \mathbb{R}^d$, where $(W^r_t)_{r=1}^{\infty}$ sequence of independent $\mathcal{F}_t$-Wiener martingales on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$, $\mathcal{L}_t$ and $\mathcal{M}_t$ are second and first order differential operators.

- L_2-theory, uniform stochastic parabolicity

$$
\mathcal{L}_t\varphi(x) = D_\alpha(a^{\alpha\beta}_t(x)D_\beta\varphi(x)), \quad \mathcal{M}^r_t\varphi(x) = b^{\alpha r}_t D_\alpha\varphi(x),
$$

$\alpha \in \{0, 1, ..., d\}$, $D_i = \partial/\partial x^i$ for $\alpha = i \neq 0$, $D_0 = I$.

Notation: K constant, m integer, H^m the closure of $C_0^\infty(\mathbb{R}^d)$ in the norm $|\varphi|_m = |(I-\Delta)^{m/2}|_{L_2}$; $H^m(\ell_2)$ is the H^m-space of ℓ_2-valued functions defined on $\mathbb{R}^d$, $H^m = W_2^m$.

Assumption 1. $m \geq 0$. For each α, β the derivatives in x of $a^{\alpha\beta}$ and of $(b^{\alpha r})_{r=1}^\infty$ up to order m are $\mathcal{P} \otimes \mathcal{B}([0,T] \times \mathbb{R}^d)$-measurable functions with values in $\mathbb{R}$ and in ℓ_2, respectively, in magnitude bounded by K.

Assumption 2. ψ is H^m-valued $\mathcal{F}_0$-measurable, $f = f_t$ and $g = (g_t^r)_{r=1}^\infty$ are adapted processes with values in H^m and in $H^m(\ell_2)$, respectively, such that

$$\mathcal{K}_T^2(|f|_{m-1}, |g|_m) := \int_0^T |f_t|_{m-1}^2 + |g_t|_m^2 \, dt < \infty.$$

Def. An L_2-valued continuous adapted process $u = (u_t)_{t\in[0,T]}$ is a generalised solution to (1)-(2) if $u \in L_2([0,T], H^1)$ and a.s.

$$(u_t, \varphi) = (\psi, \varphi) + \int_0^t (a_s^{\alpha\beta} D_\beta u_s, D_\alpha^* \varphi) + (f_s, \varphi)\, ds$$

$$+ \int_0^t (b_s^{\alpha r} D_\alpha u_s + g_s^r, \varphi)\, dW_s^r.$$

Assumption 3. There is a constant $\kappa > 0$ such that $P \otimes dt \otimes dx$-a.e.

$$\sum_{i,j=1}^d a_t^{ij}(x) z^i z^j - \frac{1}{2} \sum_{i,j=1}^d b_t^{ik}(x) b_t^{jk}(x) z^i z^j \geq \kappa \sum_{i=1}^m |z^i|^2 \quad \text{for all } z \in \mathbb{R}^d.$$

for all $t \in [0,T]$ and $\varphi \in C_0(\mathbb{R}^d)$.

Theorem 1. *Let Assumptions 1-3 hold. Then there is a unique generalised solution* $u = (u_t)_{t\in[0,T]}$ *to* (1)-(2). *Moreover,* $u \in C([0,T], H^m) \cap L_2([0,T], H^{m+1})$, *and there is a constant* $N = N(K, d, T, \kappa, m)$ *such that*

$$E \sup |u_t|_m^2 + E\int_0^T |u_t|_{m+1}^2\,dt \le N\{E|\psi|_m^2 + E\mathcal{K}_T^2(|f|_{m-1}, |g|_m)\}. \tag{3}$$

Proof. Let $\langle , \rangle$ denote the duality pairing between H^1 and H^{1*}.

$$|(a^{\alpha\beta}D_\beta v, D_\alpha\varphi)| \le N|v|_1|\varphi|_1, \quad \sum_r |(b^{\alpha r}D_\alpha v, \varphi)|^2 \le N|v|_1^2|\varphi|_0^2$$

for $v, \varphi \in C_0^\infty$. Hence for $V := H^1$, $H := L_2$, ψ, $f = (f_t)_{t\in[0,T]}$, $g = (g_t)_{t\in[0,T]}$, and $\mathbb{A}_t : V \to V^*$, $\mathbb{B}_t^r : V \to H^*$ defined by

$$\langle \mathbb{A}_t v, \varphi\rangle := (a_t^{\alpha\beta}D_\beta v, D_\alpha\varphi), \quad (\mathbb{B}_t^r v, \varphi) := (b_t^{\alpha r}D_\alpha v, \varphi)$$

we have $V \hookrightarrow H \equiv H \hookrightarrow V^*$, and

(i) $\langle \mathbb{A} v, \varphi \rangle$, $(\mathbb{B}^r v, \varphi)$ are predictable processes for $v, \varphi \in V$,

$$|\langle \mathbb{A} v, \varphi \rangle| \leq N |v|_V |\varphi|_V, \quad \sum_r |(\mathbb{B}^r v, \varphi)|^2 \leq N |v|_V^2 |\varphi|_H^2;$$

(ii) $\psi \in H$, $f \in L_2([0,T], V^*)$, $g \in L_2([0,T], H(\ell_2))$;

(iii) $2\langle \mathbb{A}_t v, v \rangle + \sum_r |\mathbb{B}_t^r v|_H^2 + \kappa |v|_V^2 \leq N |v|_H^2, \quad v \in V.$

From (i)-(iii) we get the existence of a unique solution $u \in C([0,T], H) \cap L_2([0,T], V)$ to

$$du(t) = (\mathbb{A}_t u_t + f_t)\, dt + (\mathbb{B}_t^r u_t + g_t^r)\, dW_t^r, \quad u_0 = \psi,$$

i.e., there is a unique generalised solution u to (1)-(2), and it satisfies (3).

To prove $u \in C([0,T], H^m) \cap L_2([0,T], H^{m+1})$ now take $V := H^{m+1}$, $H := H^m$, use $(,)$ for the inner product $(,)_m$ in H, and $\langle , \rangle$ for the duality pairing between V and V^*, and notice that

$$|(a^{\alpha\beta} D_\beta v, D_\alpha \varphi)_m| \le N|v|_{m+1}|\varphi|_{m+1}, \quad \sum_r |(b^{r\alpha}, \varphi)_m|^2 \le N|v|^2_{m+1}|\varphi|^2_m$$

Then $V \hookrightarrow H \equiv H \hookrightarrow V^*$, and for $\mathbb{A}_t : V \to V^*$, $\mathbb{B}^r_t : V \to H$ defined by

$$\langle A_t v, \varphi \rangle = (a^{\alpha\beta}_t D_\beta v, D^\alpha \varphi)_m, \quad (B^r_t v, \varphi) = (b^{\alpha r} D_\alpha v, \varphi)_m, \quad v, \varphi \in V$$

(i) and (iii) hold. Clearly, (ii) also holds for ψ, f and g.

Thus there is a unique solution $v \in C([0,T],H) \cap L_2([0,T],V)$ to

$$dv_t = (\mathbb{A}_t v_t + f_t)\,dt + (\mathbb{B}^r_t v_t + g^r_t)\,dW^r_t, \quad v_0 = \psi,$$

i.e., a.s.,

$$(v_t,\varphi)_m = (\psi,\varphi)_m + \int_0^t \langle \mathbb{A}_s v_s + f_s, \varphi\rangle\,ds + \int_0^t (\mathbb{B}^r_s v_s + g^r_s, \varphi)_m\,dW^r_s$$

for all $t \in [0,T]$ and $\varphi \in C_0^\infty$, and estimate (3) holds for v.

Hence with $\Lambda := (I - \Delta)^{1/2}$, a.s.,

$$(v_t, \Lambda^{2m}\varphi)_0 = (\psi, \Lambda^{2m}\varphi)_0 + \int_0^t (a_s^{\alpha\beta} D_\beta v_s, D_\alpha^* \Lambda^{2m}\varphi)_0 \, ds$$

$$+ \int_0^t (b_s^{\alpha r} v_s + g_s^r, \Lambda^{2m}\varphi)_0 \, dW_s^r \quad t \in [0, T], \, \varphi \in C_0^\infty.$$

Consequently, u is a generalised solution to (1)-(2), i.e., $v = u$. $\square$

Definition. A random field $u = \{u_t(x) : t \in [0, T], x \in \mathbb{R}^d\}$ is a classical solution to (1)-(2) if

(i) The derivatives in x of u up to second order are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable, and continuous in (t, x) for each ω.

(ii) For each multi-index $|\gamma| \leq 2$

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} |D^\gamma u_t(x)| < \infty$$

(iii) For all $\omega \in \Omega$, $t \in [0, \infty)$, $x \in \mathbb{R}^d$

$$u_t(x) = u_0(x) + \int_0^t D_\alpha(a_s^{\alpha\beta} u_s(x)) + f_s(x)\, ds$$

$$+ \int_0^t (b_s^{\alpha r} D_\alpha u_s(x) + g_s^r(x))\, dW_s^r$$

for an appropriate modification of the stochastic integral in the right-hand side.

Theorem. Let Assumptions 1-3 for the existence of a unique generalised solution hold with $m \geq 2 + d/2$. Then there is a unique classical solution to (1)-(2).

This theorem follows from the above existence and unique theorem for the generalised solution, and from the Sobolev theorem on embedding W_2^m into C_b^n for $m > n + d/2$.

- L_2-theory, (degenerate) stochastic parabolicity

$$du_t(x) = (L_t u_t(x) + f_t(x))\, dt + (M_t^r u_t(x) + g_t^r(x))\, dW_t^r, \tag{4}$$
$$u_0(x) = \psi(x) \tag{5}$$

$$L_t := a_t^{ij}(x) D_i D_j + b_t^i(x) D_i + c_t(x), \quad M_t^r := \sigma_t^{ir}(x) D_i + \nu_t^r(x)$$

Assumption (a). Derivatives in x of a^{ij} up to order $\max(m, 2)$, of b^i and c up to order m, are $\mathcal{P} \otimes \mathcal{B}([0, T] \times \mathbb{R}^d$-measurable functions, bounded by K. The derivatives in x of $\sigma^i = (\sigma^{ir})_{r=1}^{\infty}$, $\nu = (\nu^r)_{r=1}^{\infty}$ up to $m+1$ are ℓ_2-valued $\mathcal{P} \otimes \mathcal{B}([0, T] \times \mathbb{R}^d$-measurable functions, bounded by K.

Assumption (b) ψ is H^m-valued $\mathcal{F}_0$-measurable; $f = (f_t)_{t\in[0,T]}$ and $g = ((g_t^r)_{r=1}^{\infty})_{t\in[0,T]}$ are adapted processes with values in H^m and in H^{m+1}, respectively, such that

$$\int_0^T |f_t|_m^2 + |g_t|_{m+1}^2 \, dt < \infty \quad a.s.$$

Def. An H^1-valued weekly continuous adapted process is a generalised solution to (4)-(5) if a.s.

$$(u_t, \varphi) = (\psi, \varphi) + \int_0^t -(a_s^{ij} D_i u_s, D_j \varphi) + (\bar{b}_s^i D_i v_s + c_s + f_s, \varphi) \, ds$$

$$+ \int_0^t (\sigma^{ir} D_i v_s + \nu_s^r, \varphi) \, dW_s^r$$

for all $t \in [0, T]$, $\varphi \in C_0^\infty$, where $\bar{b}^i := b^i - D_j a^{ij}$.

Define $\tilde{a}^{ij} := a^{ij} - \frac{1}{2}b^{ir}b^{jr}$, $i, j = 1, 2, ..., d$.

Assumption (c) $(\tilde{a}^{ij}) \geq 0$ $P \otimes dt \otimes dx$ a.e.

Theorem 2. *Let Assumptions (a), (b) and (c) hold with $m \geq 1$. Then there is a unique generalised solution u to (4)-(5). Moreover,*

$$u \in C_w([0, T], H^m) \cap C([0, T], H^{m-1}),$$

and there is a constant $N = N(K, T, d, m)$ such that

$$E \sup_{t \in [0,T]} |u_t|_k^2 \leq N\{E|\psi|_k^2 + E\mathcal{K}_T^2(|f|_k, |g|_{k+1})\} \tag{6}$$

for every $k = 0, 1, ..., m$.

Proof. We may assume $E|\psi|_m^2 + E\mathcal{K}_T^2(|f|_m + |g|_{m+1}^2) < \infty$. The proof is based on the following estimate.

Lemma 1. Define for $u \in H^{m+2}$, $f \in H^m$, $g \in H^{m+1}(\ell_2)$

$$I_t(k,u,f,g) := \sum_{|\gamma|\le k} \{2(D^\gamma u, D^\gamma(a^{ij}u_{ij} + b^i u_i + cu + f))_0$$

$$+ \sum \sum_r |D^\gamma(\sigma^{ir}u_i + \nu^r u + g^r)|_0^2\},$$

$$J_t(u,g) := \sum_{|\gamma|\le k} \sum_r |(D^\gamma u, D^\gamma(\sigma^{ir}u_i + \nu^r iu_i + cu + f))_0|^2$$

where $u_i := D_i$, $u_{ij} := D_i D_j u$, $k = 0, 1, ..., m$. Then there is $N = N(m, K, d)$ such that

$$I_t(u,f,g) \le N(|u|_k^2 + |f|_k^2 + |g|_{k+1}^2), \quad J_t(u,g) \le N(|u|_k^4 + |u|_k^2 |g|_k^2).$$

- Existence

regularise the data: for $\varepsilon \in (0,1)$ define

$$(a^{\varepsilon ij}) := (a^{(\varepsilon)ij} + \varepsilon I, \quad b^{(\varepsilon)i},\, c^{(\varepsilon)},\, f^{(\varepsilon)}, \sigma^{(\varepsilon)ir},\, \nu^{(\varepsilon)r},\, g^{(\varepsilon)r},\, \psi^{(\varepsilon)},$$

where $h^{(\varepsilon)} := h * \rho_\varepsilon$, $\rho_\varepsilon = \varepsilon^{-d} h(\cdot/\varepsilon)$, $\rho \in C_0^\infty(\mathbb{R}^d)$, $(\rho, 1) = 1$.

Then

$$2a^{\varepsilon ij} z^i z^j - \sigma^{(\varepsilon)ir} \sigma^{(\varepsilon)jr} z^i z^j \geq \varepsilon |z|^2, \quad z = (z^1, ..., z^d) \in \mathbb{R}^d$$

Hence by Theorem 1 problem (4)-(5) with the regularised data has a unique generalised solution u^ε such that

$$u^\varepsilon \in C([0,T], H^n) \quad \text{for every } n.$$

Take $k \le m$. By Itô's formula for $|D^\gamma u^\varepsilon|_0^2$ we get

$$|u^\varepsilon|_k^2 = \sum_{|\gamma|\le k} |D^\gamma \psi^{(\varepsilon)}|_0^2 + \int_0^t I_s^\varepsilon(u_s^\varepsilon, f_s^{(\varepsilon)}, g_s^{(\varepsilon)})\, ds + \xi_t,$$

where I^ε is defined as I, but with the regularised data, and

$$\xi_t := \sum_{|\gamma|\le k} \int_0^t 2D^\gamma u_s^\varepsilon D^\gamma(\sigma_s^{ir} D_i u_s^\varepsilon + g_s^\varepsilon)\, dW_s^r.$$

By Lemma 1 with $N = N(K, m, d)$

$$|u_t^\varepsilon|_k^2 \le |\psi|_k^2 + N \int_0^t |u_s^\varepsilon|_k^2 + |f_s|_k^2 + |g_s|_{k+1}^2\, ds + \xi_t. \qquad (7)$$

Taking expectation and using Gronwall's lemma

$$E|u_t^\varepsilon|_k^2 \le N(E|\psi|_k^2 + E\int_0^T |f_s|_k^2 + |g_s|_{k+1}^2\, ds).$$

From (7)

$$E\sup_{t\le T}|u_t^\varepsilon|_k^2 \le E|\psi|_k^2 + NE\int_0^T |f_s|_k^2 + |g_s|_{k+1}^2\, ds + E\sup_{t\le T}|\xi_t|.$$

By the Davis inequality $E\sup_{t\le T}|\xi_t| \le 3E[\xi]_T^{1/2}$

$$\le 6\sum_{|\gamma|\le k} E\left(\int_0^T \sum_r |(D^\gamma u^\varepsilon, D^\gamma(\sigma^{(\varepsilon)ir}D_i u^\varepsilon + \nu^\varepsilon + g^{(\varepsilon)r}))_0|^2\, ds\right)^{1/2}$$

$$\le NE(\int_0^T J_s^\varepsilon(k, u_s^\varepsilon, g_s^{(\varepsilon)})\, ds)^{1/2} \le NE(\int_0^T |u_s^\varepsilon|_k^4 + |u_s^\varepsilon|_k^2 |g_s|_{k+1}^2\, ds)^{1/2}$$

Note

$$E\Big(\int_0^T |u_s^\varepsilon|_k^4 + |u_s^\varepsilon|_k^2||g_s|_{k+1}^2\,ds\Big)^{1/2} \le A + B,$$

where

$$A := E\Big(\int_0^T |u_s^\varepsilon|_k^4\,ds\Big)^{1/2} \le \frac{1}{4}E\sup_{t\le T}|u_s^\varepsilon|_k^2 + E\int_0^T |u_s^\varepsilon|_k^2\,ds \quad ,$$

and

$$B := E\Big(\int_0^T |u_s^\varepsilon|_k^2|g_s|_{k+1}^2\,ds\Big)^{1/2} \le \frac{1}{4}E\sup_{t\le T}|u_s^\varepsilon|_k^2 + E\int_0^T |g_s|_{k+1}^2\,ds.$$

Summing up

$$E \sup_{t \le T} |u_s^\varepsilon|_k^2 \le N(E|\psi|_k^2 + E \int_0^T |f_s|_k^2 + |g_s|_{k+1}^2 \, ds) + \frac{1}{2} E \sup_{t \le T} |u_s^\varepsilon|_k^2,$$

which gives

$$E \sup_{t \le T} |u_s^\varepsilon|_k^2 \le NE|\psi|_k^2 + NE \int_0^T |f_s|_k^2 + |g_s|_{k+1}^2 \, ds.$$

For $r>1$ let $\mathbb{H}^n_{2,r}$ denote the space of H^n-valued predictable processes $v=(v_t)_{t\in[0,T]}$ such that

$$|v|^2_{\mathbb{H}^n_{2,r}} := E\left(\int_0^T |v_t|_n^r\,dt\right)^{2/r} < \infty.$$

Then for all $n=0,1,...,m$

$$|u^\varepsilon|^2_{\mathbb{H}^n_{2,r}} \le T^{1/r} E \sup_{t\le T} |u^\varepsilon|^2_n \le NE|\psi|^2_n + NE\int_0^T |f_s|^2_n + |g_s|^2_{n+1}\,ds \le C.$$

with a constant C, independent of ε. This means $\{u^\varepsilon : \varepsilon\in(0,1)\}$ is a bounded set in $\mathbb{H}^n_{2,r}$ for $r>1$ and $n=0,1,...,m$.
Hence there is a sequence $\varepsilon_k\to 0$ such that

$$v^k := u^{\varepsilon_k} \to u, \quad \text{weakly in } \mathbb{H}^n_{2,r}$$

for every integer $r>1$ and $n=0,1,...,m$.

Since $|u|_{\mathbb{H}^n_{2,r}} \leq \liminf_k |v^k|_{\mathbb{H}^n_{2,r}}$, we have

$$E\left(\int_0^T |u_t|_k^r\,dt\right)^{2/r} \leq NE|\psi|_k^2 + NE\int_0^T |f_s|_k^2 + |g_s|_{k+1}^2\,ds.$$

Letting $r \to \infty$ we get

$$E\text{ess sup}_{t\leq T}|u_t|_n^2 \leq N|\psi|_n^2 + NE\int_0^T |f_s|_n^2 + |g_s|_{n+1}^2\,ds. \quad (8)$$

Letting $k \to \infty$ in the equation for $v^k := u^{\varepsilon_k}$, we get that u is a generalised solution, i.e.,

$$(u_t,\varphi)_0=(\psi,\varphi)_0+\int_0^t -(a_s^{ij}D_iu_s,D_j\varphi)_0+(\bar{b}_s^iD_iu_s+c_su_s+f_s,\varphi)_0\,ds$$

$$+\int_0^t(\sigma_s^{ir}D_iu_s+\nu_s^ru_s+g_s^r,\varphi)_0\,dW_s^r$$

Substituting here $\Lambda^{2m-2}\varphi=(I-\Delta)^{m-1}\varphi$ in place of φ,

$$(u_t,\varphi)_m=(\psi,\varphi)_m+\int_0^t\langle F_s,\varphi\rangle\,ds+\int_0^t(G_s^r,\varphi)_m\,dW_s^r,\quad \varphi\in C_0^\infty$$

with appropriate F, G, by the theorem on Itô's formula for $|u_t|_{m-1}^2$ in

$$H^m\hookrightarrow H^{m-1}\equiv H^{m-1*}\hookrightarrow H^{m*},$$

u has an H^{m-1}-valued continuos modification.

Substituting $\Lambda^{2m}\varphi$ in place of φ we see that

$$(u_t, \Lambda^{2m}\varphi)_0 = (u_t, \varphi)_m \quad \text{is continuos in } t \text{ for every } \varphi \in C_0^\infty.$$

Hence u is weakly continuous as an H^m-valued process, which by the estimate (8) for $E\text{ess}\sup_t \leq |u_t|_n$ gives the sup estimate (6).

• Uniqueness

By virtue of the sup estimate (6), applied to the difference $w := u - v$ of generalised solutions u, v, we have $E\sup_{t\leq T}|w|_0^2 = 0$, which implies $u = v$ (a.s.). □