INTRODUCTION TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS BANGALORE, DECEMBER 2015 MANUSCRIPT

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Contents

1. Lecture 1: Sobolev spaces and interpolation

1.1. Sobolev spaces, imbedding and trace theorems.

Suppose that $\mathrm{D} \subset \mathbb{R}^d$ is an open set.

Definition 1.1. If $u, v \in L^1_{\mathrm{loc}}(\mathrm{D})$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, then

$$D^\alpha u = v \text{ in the weak sense}$$

iff $\forall\, \phi \in C_0^\infty(\mathrm{D})$

$$\int u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int v\, \phi\, dx$$

where $|\alpha| = \sum_{i=1}^d \alpha_i$.

Remark 1.2. If also $u \in C^1(\mathrm{D})$, $|\alpha| \le n$ and v is the classical partial derivative of u

$$v(x) = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_d} x_d} u$$

then $v = D^\alpha u$.

Lemma 1.3. *If u, v, $w \in L^1(\mathrm{D})_{\mathrm{loc}}$, $\alpha \in \mathbb{N}^d$, $D^\alpha u = v$ and $D^\alpha u = w$ in the weak sense then*

$$v = w \quad \textit{Lebesgue a.e.}$$

Example 1.4. If $\mathrm{D} = (-1, 1)$ and $u(x) = |x|$,

$$v(x) = \begin{cases} 1, & x \in (0,1) \\ -1, & x \in (-1,0) \end{cases}$$

$v(0)$ is any number, then $D\,u = v$ in the weak sense.

Similarly if

$$u(x) = x^+ = \begin{cases} x, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

$x \in D$, then $D\,u = v$ where

$$v(x) = \begin{cases} 1, & x \in [0,1), \\ 0, & x \in (-1,0). \end{cases}$$

Solution. *Use definition and integration by parts.*

Exercise 1.5. (**I**) If $u \in C(a,c), b \in (a,c), v \in L^1_{\text{loc}}(a,c)$ and

$$D\,u = v \ \text{ weakly on } (a,b),$$

$$D\,u = v \ \text{ weakly on } (b,c),$$

then $D\,u = v$ weakly on (a,c).

(**II**) If $D^\alpha u_1 = v_1, D^\alpha u_2 = v_2, \lambda_1, \lambda_2 \in \mathbb{R}$ then $D^\alpha(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 v_1 + \lambda_2 v_2$.

Definition 1.6. Let $p \in [1,\infty], k \in \mathbb{N}$. We say that $u \in W^{k,p}(\mathrm{D})$ iff $u \in L^p(\mathrm{D})$ and $\forall\ \alpha \in \mathbb{N}^d, |\alpha| \le k, \exists\, v \in L^p(\mathrm{D})$ such that

$$D^\alpha u = v, \quad \text{weakly.}$$

That is $u \in W^{k,p}(\mathrm{D})$ iff $\forall\ \alpha \in \mathbb{N}^d : |\alpha| \le k$,

$$D^\alpha u \in L^p(\mathrm{D}) \ \text{ in the weak sense.}$$

Note If $\alpha = (0,0,\cdots,0)$,then $D^\alpha u = u$.

Notation 1.7. $H^k(\mathrm{D}) = W^{k,2}(\mathrm{D})$.

Definition 1.8. For $u \in W^{k,p}(\mathrm{D})$ we put, $p < \infty$,

$$\begin{aligned}
\|u\|_{W^{k,p}} &= \left(\sum_{|\alpha|\le k} \int_{\mathrm{D}} |D^\alpha u(x)|^p \, dx \right)^{1/p} \\
&= \left(\sum_{|\alpha|\le k} |D^\alpha u|^p_{L^p(\mathrm{D})} \right)^{1/p}
\end{aligned}$$

and for $p = \infty$, we put

$$\begin{aligned}
\|u\|_{W^{k,\infty}} &= \sum_{|\alpha|\le k} |D^\alpha u|_{L^\infty(\mathrm{D})} \\
&= \sum_{|\alpha|\le k} \operatorname*{ess\,sup}_{x\in\mathrm{D}} |D^\alpha u(x)|.
\end{aligned}$$

Exercise 1.9. Show that the norm $\|\cdot\|_{W^{k,p}}$ is equivalent to

$$\sum_{|\alpha|\le k} |D^\alpha u|_{L^p(\mathrm{D})}.$$

Proposition 1.10. $W^{k,p}(\mathrm{D})$ *with* $\|\cdot\|_{W^{k,p}(\mathrm{D})}$ *is a normed vector space. In fact,* $W^{k,p}(\mathrm{D})$ *is a Banach space.*

Definition 1.11. $W^{k,p}_0(\mathrm{D})$ is the closure of C_0^∞ in $W^{k,p}(\mathrm{D})$, where

$$C_0^\infty(\mathrm{D}) = \{\phi : \mathrm{D}\mathbb{R} \text{ of } C^\infty \text{ class such that } \operatorname{supp}\phi \subset \mathrm{D} \text{ is compact}\}.$$

Exercise 1.12. Let $\mathrm{D} = \{x \in \mathbb{R}^d : |x| < 1\}, \alpha > 0$,

$$u(x) = |x|^{-\alpha}, \quad x \in \mathrm{D}/\{0\},$$

$u(0)$ is any number, then

$$D_i^\alpha u(x) = \frac{-\alpha x_i}{|x|^{\alpha+2}}, \quad x \in \mathrm{D}/\{0\},$$

where $D_i u = D^\alpha u$, $\alpha = (0, \cdots, 1, \cdots, 0)$, 1 appears at the i^{th} position, and

$$u \in W^{1,p}(\mathrm{D}) \text{ iff } p < \frac{d}{1+\alpha}.$$

Solution. ***Hint*** *$u \in L^p(\mathrm{D})$ iff $\int_0^1 \frac{r^{d-1}}{r^{\alpha p}}\, dr < \infty$ iff $d - p\alpha > 0$ iff $\alpha < \frac{d}{p}$.*

Proposition 1.13. ***Elementary properties***
Let $u, v \in W^{k,p}(\mathrm{D})$, $|\alpha| + |\beta| \le k$.
(i) $D^\alpha u \in W^{k-|\alpha|,p}(\mathrm{D})$,

$$D^\beta D^\alpha u = D^\alpha D^\beta u = D^{\alpha+\beta} u.$$

(ii) If $\mathrm{D}_2 \subset \mathrm{D}$ is an open set then $u \in W^{k,p}(\mathrm{D}_2)$, i.e. $u|_{\mathrm{D}_2} \in W^{k,p}(\mathrm{D}_2)$.
(iii) If $\mathrm{D}_2 \subset \mathrm{D}$ is an open set and $u \in W_0^{k,p}(\mathrm{D}_2)$ then $u \in W_0^{k,p}(\mathrm{D})$.

Theorem 1.14. ***Friedman Theorems 6.3, 7.1; Evans Theorem 2 in Section 5.3***
If D is a bounded open subset of $\mathbb{R}^d$, $1 \le p < \infty$ and $u \in W^{k,p}(\mathrm{D})$, then $\exists \{u_n\} \subset C^\infty(\mathrm{D}) \cap W^{k,p}(\mathrm{D})$ such that

$$u_n \to u \text{ in } W^{k,p}(\mathrm{D}).$$

Moreover if $\partial\mathrm{D}$ is of C^j-class, then $C^j(\bar{\mathrm{D}}) \cap W^{k,p}(D)$ is dense in $W^{k,p}(D)$.

Remark. **Evans, Theorem 3, Section 5.3**
If $\partial\mathrm{D}$ is C^1, then $C^\infty(\bar{\mathrm{D}}) \cap W^{k,p}(\mathrm{D})$ is dense in $W^{k,p}(\mathrm{D})$.

Theorem 1.15. *If D is bounded open and $\partial\mathrm{D}$ is of C^1-class and V is an open subset of $\mathbb{R}^d$ such that $\bar{\mathrm{D}} \subset V$, then*

$$\exists\, E\colon W^{1,p}(\mathrm{D}) \to W^{1,p}(\mathbb{R}^d),$$

linear and bounded, such that $\forall\, u \in W^{1,p}(\mathrm{D}), E(u) = u$, a.e. in D and $\operatorname{supp} E \subset V$.

Remark. The above theorem holds true for $W^{2,p}$ too.

Proposition 1.16. *If $\mathrm{D} = \mathbb{R}^d$ then $u \in W^{k,2}(\mathbb{R}^d)$ iff*

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 \, d\xi \; < \; \infty,$$

where $\hat{u} = \mathcal{F} u$

$$\hat{u}(\xi) = C \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\, dx,$$

$\xi \in \mathbb{R}^d$ is the Fourier transform of u.

Remark. If $s \ge 0$, then $u \in H^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ iff

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \; < \; \infty.$$

Corollary 1.17. *A map* $C_0^\infty(\mathbb{R}^d) \ni u \mapsto u|_{R^{d-1}} \in C_0^\infty(\mathbb{R}^{d-1})$ *extends to a linear bounded map from* $H^s(\mathbb{R}^d)$ *t* $H^{s-1/2}(\mathbb{R}^{d-1})$*, provided* $s > \frac{1}{2}$.

Theorem 1.18. *If* D *is bounded open set of* $\mathbb{R}^d$ *and* ∂D *is of* C^1*-class or Lipschitz , then*

$$\exists\,!\,\gamma_0\colon W^{1,p}(\mathrm{D}) \to L^p(\partial\mathrm{D}),$$

linear and bounded, such that $\gamma_0(u) = u$ *if* $u \in W^{1,p}(\mathrm{D}) \cap C(\bar{\mathrm{D}})$*. In fact*

$$\gamma_0\colon W^{1,p}(\mathrm{D}) \to W^{1-1/p,p}(\partial\mathrm{D}).$$

Let us define the following space,

$$E(\mathrm{D}) = \{u \in L^2(\mathrm{D}, \mathbb{R}^d : \mathrm{div} u \in L^2(\mathrm{D})\}.$$

Theorem 1.19. *Let* D *be as in the previous theorem. If* ∂D *is Lipschitz then* $\exists\,!$ *a bounded linear map*

$$\gamma_n\colon E(\mathrm{D}) \to H^{-1/2}(\partial\mathrm{D}),$$

where $H^{-1/2}(\partial\mathrm{D}) = \left[H^{1/2}(\partial\mathrm{D})\right]^*$ *such that*

$$\gamma_n(u) = u \cdot n|_{\partial\mathrm{D}} \quad \forall\, u \in C^\infty(\bar{\mathrm{D}}, \mathbb{R}^d),$$

where $n\colon \partial\mathrm{D} \to \mathbb{R}^d$ *is the unit normal vector field, and Stokes formula holds*

$$\int_{\mathrm{D}} u \cdot \nabla\phi = -\int_{\mathrm{D}} \mathrm{div} u\phi\, dx + \int_{\partial\mathrm{D}} \gamma_n u \cdot \gamma_0(\phi)\, dA,$$

$\forall\, u \in E(\mathrm{D}), \phi \in H^1(\mathrm{D}, \mathbb{R})$ *(so that* $\gamma_0(\phi) \in H^{1/2}(\partial\mathrm{D})$*).*

Remark 1.20. **Adams, Theorem 5.36**
If ∂D is of C^1-class, $p < d$ then

$$\gamma_0\colon W^{1,p} \to L^q(\partial\mathrm{D})$$

for $q \in [p, p^*]$, $p^* = \dfrac{(d-1)\,p}{d-p}$. If $p = d$ then above holds for $q \in [p, \infty)$. The fact that

$$\gamma_0\colon W^{1,p}(\mathrm{D}) \to W^{s-1/p,p}(\partial\mathrm{D}) = B^{s-1/p}_{p,p}(\partial\mathrm{D})$$

can be found in **Adams, Theorem 7.39**.

1.2. **Real interpolation spaces.** Let X and Y be two real Banach spaces with respect to norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. By $X = Y$ we mean that X and Y have the same elements and equivalent norms. By $Y \subset X$ we mean that Y is continuously embedded in X.

Definition 1.21. X and Y is said to be interpolation couple if there exists a Hausdorff topological vector space V such that Y and X are continuously embedded in V, and it is denoted by (X, Y).

In this case the intersection $X \cap Y$ is a linear subspace of V, and it is a Banach space with respect to norm

$$\|u\|_{X\cap Y} = \max\{\|u\|_X, \|u\|_Y\}.$$

Also the sum $X + Y = \{x + y \;:\; x \in X,\ y \in Y\}$ is a linear subspace of V, and it is a Banach space with respect to norm

$$\|u\|_{X+Y} = \inf_{x\in X, y\in Y, x+y=u} \|x\|_X + \|y\|_Y.$$

Let (X, Y) be interpolation couple. Assume that L^p_*, $p \geq 1$, is the space of all functions f defined from $(0, \infty)$ to $\mathbb{R}$ such that

$$\int_0^\infty |f(t)|^p \frac{dt}{t} < \infty.$$

It can be easily seen that L^p_* is a L^p space with respect to the measure $\frac{dt}{t}$ in $(0, \infty)$. In particular for $p = \infty$, $L^\infty_* = L^\infty$.

1.2.1. *K-method.*

Definition 1.22. For every $u \in X + Y$ and $t > 0$, set

$$K(t, u, X, Y) = \inf_{u=x+y, x\in X, y\in Y} (\|x\|_X + t\|y\|_Y).$$

We will use $K(t, u)$ instead of $K(t, u, X, Y)$ for ease.
Note that $K(1, u) = \|u\|_{X+Y}$, and for every $t > 0$, $K(t, u)$ is a norm in $X + Y$, which is equivalent to the norm $\|\cdot\|_{X+Y}$.

Definition 1.23. For each $\theta \in (0, 1)$ and $p \in [1, \infty]$ the following spaces

$$(X, Y)_{\theta,p} = \{u \in X + Y \;:\; t \mapsto t^{-\theta} K(t, u) \in L^p_*\}$$

and

$$(X, Y)_\theta = \{u \in X + Y \;:\; \lim_{t\to 0^+} t^{-\theta} K(t, u) = \lim_{t\to\infty} t^{-\theta} K(t, u) = 0\}$$

are called real interpolation spaces.

Proposition 1.24. *For each $\theta \in (0, 1)$ and $p \in [1, \infty]$, the following mapping*

$$(X, Y)_{\theta,p} \ni u \mapsto \|u\|_{\theta,p} = \|t^{-\theta} K(t, u)\|_{L^p_*}$$

is a norm in $(X, Y)_{\theta,p}$. Moreover, $(X, Y)_{\theta,p}$ is a Banach space with respect to this norm.

Proposition 1.25. *For each $\theta \in (0, 1)$, the following mapping*

$$(X, Y)_\theta \ni u \mapsto \|u\|_{\theta,\infty} = \|t^{-\theta} K(t, u)\|_{L^\infty_*}$$

is a norm in $(X, Y)_\theta$. Moreover $(X, Y)_\theta$ is a Banach space endowed with this norm.

Example 1.26. Let $W^{1,p}(\mathbb{R})$, $p \in [1, \infty)$, be the space of all function $f \in L^p(\mathbb{R})$ such that the first weak derivative Df of f belongs to $L^p(\mathbb{R})$.

1.2.2. *Trace method.*

Definition 1.27. Assume that $\theta \in (0,1)$ and $p \in [1,\infty]$. Define

$$\begin{aligned} V(p,\theta,Y,X) \;\; := \;\; & \Big\{ u : \mathbb{R}_+ \to X+Y : u \in W^{1,p}_{\text{loc}}((0,\infty); X+Y), \text{ and} \\ & \{ t \mapsto u_\theta(t) = t^\theta u(t) \} \in L^p_*(0,+\infty;Y), \\ & \{ t \mapsto v_\theta(t) = t^\theta u'(t) \} \in L^p_*(0,+\infty;X) \Big\} \end{aligned}$$

where for any $I \subset (0,\infty)$, $L^p_*(I)$ is the L^p Lebesgue space on I with respect to the measure $\frac{dt}{t}$. In particular,

$$|u_\theta|^p_{L^p_*(0,+\infty;Y)} = \int_0^\infty |u_\theta(t)|^p_Y \, \frac{dt}{t}.$$

The set $V(p,\theta,Y,X)$ is endowed with the norm

$$|u|^p_{V(p,\theta,Y,X)} := \int_0^\infty |t^\theta u(t)|^p_Y \, \frac{dt}{t} + \int_0^\infty |t^\theta u'(t)|^p_X \, \frac{dt}{t}$$

Proposition 1.28. *If $\theta \in (0,1)$ and $p \in [1,\infty)$ the real interpolation space $(X,Y)_{\theta,p}$ is equal to the set of all traces at $t = 0$ of all functions $u \in V(p,1-\theta,Y,X)$. Moreover, the original norm is equivalent to the following one*

$$\|u\|_{\text{tr},\theta,p} := \inf\{ \, |u|_{V(p,1-\theta,Y,X)} : x = u(0), u \in V(p,1-\theta,Y,X)\}. \tag{1.1}$$

Example 1.29. Let us choose $X = L^p(\mathbb{R}^d)$ and $Y = W^{1,p}(\mathbb{R}^d)$. Then the space $W^{1-\frac{1}{p},p}(\mathbb{R}^d) = (L^p(\mathbb{R}^d); W^{1,p}(\mathbb{R}^d))_{1-\frac{1}{p},p}$ is equal to the set of all traces $u(0)$ where $u : [0\infty) \to W^{1,p}(\mathbb{R}^d)$ belongs to L^p, u is weakly differentiable as an valued function and $u' : [0\infty) \to L^p(\mathbb{R}^d)$ also belongs to L^p, i.e.

$$\begin{aligned} \int_0^\infty |u(t)|^p_{W^{1,p}(\mathbb{R}^d)} \, dt < \infty, \\ \int_0^\infty |D_t u(t)|^p_{L^p(\mathbb{R}^d)} \, dt < \infty. \end{aligned}$$

Note that the LHS in the first condition above can be written as

$$\sum_{j=1}^d \int_0^\infty \int_{\mathbb{R}^d} D_{x_i} |u(t,x)|^p \, dx \, dt.$$

All derivatives above are understood in the weak sense.

1.2.3. *Special case: domain of a generator.* Assume that exist constants C_0 and $\omega_0 < 0$ such that

$$\|e^{-tA}\| \leq C_0 e^{t\omega_0}, \; t \geq 0. \tag{1.2}$$

Without loss of generality, we will assume from now on that Let us also recall the following characterization of the real interpolation[1] spaces $(E, D(A^m))_{\theta,q}$ =

[1]In order to fix the notation let us point out that the interpolation functor $(X_0,X_1)_{\theta,q}$, $\theta \in (0,1)$, $q \in [1,\infty]$, between two Banach spaces X_1 and X_0 such that both are continuously embedded into a common topological Hausdorff vector space, satisfies the following properties: (i)$(X_1,X_0)_{\theta,q}$ =

$(D(A^m), E)_{1-\theta,q}$, where $m \in \mathbb{N}$, between $D(A^m)$ and E with parameters $\theta \in (0,1)$ and $q \in [1,\infty)$, see section 1.14.5 in [40]:

$$(1.3) \qquad (E, D(A^m))_{\vartheta,q} = \left\{ x \in E : \int_0^\infty |t^{m(1-\vartheta)} A^m e^{-tA} x|^q \frac{dt}{t} < \infty \right\}.$$

The space $(E, D(A^m))_{\theta,q}$ is often denoted by $D_{A^m}(\theta, p)$ and u the following notation is often used

$$(1.4) \qquad \boxed{|x|^q_{D_{A^m}(\theta,q);\delta} = \int_0^\infty |t^{m(1-\theta)} A^m e^{-tA} x|^q \frac{dt}{t}} \; .$$

Important property.

Theorem 1.30. *If A is an unbounded positive selfadjoint operator in a Hilbert space H, then*

$$[H, D(A)]_\theta = (H, D(A))_{\theta,2}.$$

In particular, if H_1, H_2 are two Hilbert spaces, then

$$[H_1, H_2]_\theta = (H_1, H_2)_{\theta,2}.$$

2. Lecture 2: Elliptic equations, L^2 (i.e variational) and L^p (i.e. after Agmon Douglas and Nirenberg) theories; parabolic problems (variational approach via Lions-Magenes)

2.1. **Parabolic problems in L^p spaces.**

(via Dore-Venni and Weis), examples of "nonlinear" problems: Navier-Stokes (via Temam), Schrödinger (via Byrq et all) and heat flow (via Eells-Sampson or Struwe) (or Landau-Lifshirz, easier as the target manifold is a sphere)

3. Lecture 3: γ-radonifying operators,

3.1. **Definitions and basic properties.**

3.2. **Examples of γ-radonifying operarators.**

In [14] we formulated and in [12], we proved the following, see Theorem 2.3,

Theorem 3.1. *Suppose* H *is a separable real Hilbert space and let $p \in (1,\infty)$ be fixed. Let $(\mathcal{O}, \mathcal{F}, \nu)$ be a σ-finite measure space. For a bounded linear operator $K : \mathrm{H} \to L^p(\mathcal{O})$ the following assertions are equivalent:*

(1) *K is γ-radonifying;*

$(X_0, X_1)_{1-\theta,q}$, (ii) if $X_0 \subset X_1$, $0 < \theta_1 < \theta_2 < 1$ and $p, q \in [1,\infty]$, then $(X_0, X_1)_{\theta_1,p} \subset (X_0, X_1)_{\theta_2,q}$. Roughly speaking, (ii) implies that, if $X_0 \subset X_1$, then $(X_0, X_1)_{\vartheta,p} \searrow X_0$ as $\vartheta \searrow 0$ and $(X_0, X_1)_{\theta,p} \nearrow X_1$ as $\vartheta \nearrow 0$. Or equivalently, if $X_0 \subset X_1$, then $(X_1, X_0)_{\theta,p} \searrow X_0$ as $\theta \nearrow 1$ and $(X_1, X_0)_{\theta,p} \nearrow X_1$ as $\theta \searrow 1$. See Proposition 1.1.4 in [33] and section 1.3.3 in [40].

(2) *There exists a ν-measurable function $\kappa : \mathcal{O} \to \mathrm{H}$ with*

$$\int_{\mathcal{O}} |\kappa(x)|_{\mathrm{H}}^p \, d\nu(x) < \infty \tag{3.1}$$

such that for all ν-almost all $x \in \mathcal{O}$ we have

$$(K(h))(x) = [\kappa(x), h]_{\mathrm{H}}, \qquad h \in \mathrm{H}. \tag{3.2}$$

Moreover, there exists a constant $C > 0$ independent of K such that

$$\frac{1}{C} \int_{\mathcal{O}} |\kappa(x)|^p \, d\nu(x) \leq \|K\|_{R(Y,L^p)}^p \leq C \int_{\mathcal{O}} |\kappa(x)|^p \, d\nu(x).$$

In the special case when H is equal to $L^2(D)$ over some measure space $(D, \mathcal{C}, \mu)$, the above result reads as follows

Theorem 3.2. *Suppose that $(\mathcal{O}, \mathcal{F}, \nu)$ and $(D, \mathcal{C}, \mu)$ are a σ-finite measure spaces. Let $p \in (1, \infty)$ be fixed. For a bounded linear operator $K : L^2(D) \to L^p(\mathcal{O})$ the following assertions are equivalent:*

K is γ-radonifying;

There exists a $\nu \otimes \mu$-measurable function $\kappa : \mathcal{O} \times D \to \mathbb{R}$ with

$$\int_{\mathcal{O}} \left[\int_D |\kappa(x, y)|^2 \, d\mu(y) \right]^{p/2} d\nu(x) < \infty \tag{3.3}$$

such that for all ν-almost all $x \in \mathcal{O}$ we have

$$(K(h))(x) = \int_D \kappa(x, y) h(y) \, d\mu(y), \qquad h \in L^2(D). \tag{3.4}$$

Moreover, there exists a constant $C > 0$ independent of K such that

$$\frac{1}{C} \int_{\mathcal{O}} \left[\int_D |\kappa(x, y)|^2 \, d\mu(y) \right]^{p/2} d\nu(x) \leq \|K\|_{R(Y,L^p)}^p \leq C \int_{\mathcal{O}} \left[\int_D |\kappa(x, y)|^2 \, d\mu(y) \right]^{p/2} d\nu(x).$$

Our aim is to present various generalisation of the above Theorems. Firstly, we will describe generalisation of Theorem 3.1 when the Banach space $L^p(\mathcal{O})$ is replaced by $L^q(\mathcal{O}_1); L^p(\mathcal{O}_2))$, with $1 < p, q < \infty$, for some two σ-finite measure spaces $(\mathcal{O}_i, \mathcal{F}_i, \nu_i)$, $i = 1, 2$. We shall prove

Theorem 3.3. *Suppose* H *is a separable real Hilbert space and let $p, q \in (1, \infty)$. Let $(\mathcal{O}_i, \mathcal{F}_i, \nu_i)$, $i = 1, 2$, be σ-finite measure spaces. For a bounded linear operator $K : \mathrm{H} \to L^q(\mathcal{O}_1; L^p(\mathcal{O}_2))$ the following assertions are equivalent:*

(i) *K is γ-radonifying;*

(ii) *There exists a $\nu_1 \otimes \nu_2$-measurable function $\kappa : \mathcal{O}_1 \times \mathcal{O}_2 \to \mathrm{H}$ with*

$$\int_{\mathcal{O}_1} \left[\int_{\mathcal{O}_2} |\kappa(x_1, x_2)|_{\mathrm{H}}^p \, d\nu_2(x_2) \right]^{q/p} d\nu_1(x_1) < \infty \tag{3.5}$$

such that for all $\nu_1 \otimes \nu_2$-almost all $(x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2$ we have

$$(K(h))(x_1, x_2) = [\kappa(x_1, x_2), h]_{\mathrm{H}}, \qquad h \in \mathrm{H}. \tag{3.6}$$

Moreover, there exists a constant $C > 0$ independent of K such that

$$\frac{1}{C}\int_{O_1}\left[\int_{O_2}|\kappa(x_1,x_2)|^p_{\mathrm{H}}\,d\nu_2(x_2)\right]^{q/p}\,d\nu_1(x_1)$$
$$\leq \quad \|K\|^p_{R(Y,L^p)} \leq C\int_{O_1}\left[\int_{O_2}|\kappa(x_1,x_2)|^p_{\mathrm{H}}\,d\nu_2(x_2)\right]^{q/p}\,d\nu_1(x_1).$$

When in Theorem 3.3 we choose $\mathrm{H} = L^2(D_1); L^p(D_2))$ for another pair of σ-finite measure spaces $(D_i, \mathcal{C}_i, \mu_i)$, $i = 1, 2$ we get the following results which is a natural generalisation of Theorem 3.2.

Theorem 3.4. *Assume that $(D_i, \mathcal{C}_i, \mu_i)$, $i = 1, 2$ and $(O_i, \mathcal{F}_i, \nu_i)$, $i = 1, 2$, be σ-finite measure spaces. Let $p, q \in (1, \infty)$.*
For a bounded linear operator $K : L^2(D_1; L^2(D_2)) \to L^q(O_1; L^p(O_2))$ the following assertions are equivalent:

(1) *K is γ-radonifying;*
(2) *There exists a $\nu \otimes \mu$-measurable function $\kappa : O \times D \to \mathbb{R}$ with*
(3.7)
$$\int_{O_1}\left\{\int_{O_2}\left[\int_{D_1\times D_2}|\kappa(x_1,x_2;y_1,y_2)|^2\,d(\mu_1\otimes\mu_2)(y_1,y_2)\right]^{p/2}\,d\nu_2(x_2)\right\}^{q/p}\,d\nu_1(x_1) < \infty$$

such that for any $h \in L^2(D_1; L^2(D_2))$, for all $\nu_1 \otimes \nu_2$-almost all $(x_1, x_2) \in O_1 \times O_2$ we have

$$(K(h))(x_1,x_2) = \int_{D_1\times D_2}\kappa(x_1,x_2;y_1,y_2)h(y)\,d(\mu_1\otimes\mu_2)(y_1,y_2). \tag{3.8}$$

Moreover, there exists a constant $C > 0$ independent of K such that

$$\frac{1}{C}\int_{O_1}\left\{\int_{O_2}\left[\int_{D_1\times D_2}|\kappa(x_1,x_2;y_1,y_2)|^2\,d(\mu_1\otimes\mu_2)(y_1,y_2)\right]^{p/2}\,d\nu_2(x_2)\right\}^{q/p}\,d\nu_1(x_1)$$
$$\leq \quad \|K\|^p_{R(Y,L^p)}$$
$$\leq C\int_{O_1}\left\{\int_{O_2}\left[\int_{D_1\times D_2}|\kappa(x_1,x_2;y_1,y_2)|^2\,d(\mu_1\otimes\mu_2)(y_1,y_2)\right]^{p/2}\,d\nu_2(x_2)\right\}^{q/p}\,d\nu_1(x_1).$$

Proof of Theorem 3.3. Let $(e_j)_{j=1}^\infty$ be the ONB of the Hilbert space H. Let $(\beta_j)_{j=1}^\infty$ be a sequence of i.i.d. standard mean 0 gaussian random variables. Then by the Fubini, Kahane and Plancherel Theorems for any $n \in \mathbb{N}$ we have

$$
\begin{aligned}
&\mathbb{E}\int_{O_1}\left[\int_{O_2}|\sum_j^n \beta_j K(e_j)(x_1,x_2)|^p\,d\nu_2(x_2)\right]^{q/p}\,d\nu_1(x_1)\\
=\;&\int_{O_1}\mathbb{E}\left[\int_{O_2}|\sum_j^n \beta_j K(e_j)(x_1,x_2)|^p\,d\nu_2(x_2)\right]^{q/p}\,d\nu_1(x_1)\\
=\;&\int_{O_1}\mathbb{E}\|\sum_j^n \beta_j K(e_j)(x_1,\cdot)\|^q_{L^p(O_2,\nu_2)}\,d\nu_1(x_1)\\
\sim\;&\int_{O_1}\left[\mathbb{E}\|\sum_j^n \beta_j K(e_j)(x_1,\cdot)\|^p_{L^p(O_2,\nu_2)}\right)^{q/p}\,d\nu_1(x_1)\\
=\;&\int_{O_1}\left[\mathbb{E}\int_{O_2}|\sum_j^n \beta_j K(e_j)(x_1,x_2)\,d\nu_2(x_2)|^p\right)^{q/p}\,d\nu_1(x_1)\\
=\;&\int_{O_1}\left[\int_{O_2}\mathbb{E}|\sum_j^n \beta_j K(e_j)(x_1,x_2)\,d\nu_2(x_2)|^p\right)^{q/p}\,d\nu_1(x_1)\\
=\;&c_p^{q/p}\int_{O_1}\left[\int_{O_2}\left(\sum_j^n |K(e_j)(x_1,x_2)|^2\right)^{p/2}\,d\nu_2(x_2)|^p\right)^{q/p}\,d\nu_1(x_1)\\
=\;&c_p^{q/p}\int_{O_1}\left[\int_{O_2}\left(|\kappa(x_1,x_2)|^2\right)^{p/2}\,d\nu_2(x_2)|^p\right)^{q/p}\,d\nu_1(x_1)\\
=\;&c_p^{q/p}\int_{O_1}\left[\int_{O_2}|\kappa(x_1,x_2)|^p\,d\nu_2(x_2)|^p\right)^{q/p}\,d\nu_1(x_1).
\end{aligned}
$$

The result then follows by applying Itô-Nisio Theorem, see [32].

□

3.2.1. *Applications.* We will apply Theorem 3.4 with $D_1 = O_1 = (0,T)$ and $D_2 = O_2 = D$, where $T \in (0,\infty)$ and $D \subset \mathbb{R}^d$ and where $\nu_1 = \mu_1 =$ the Lebeuge measure on $(0,T)$ and $\nu_2 = \mu_2 =$ the Lebeuge measure on D. Suppose that A is a generator of a C_0 semigroup on $L^2(D)$. For $\alpha,\beta \in (0,1)$ we consider the operator $K_{\alpha,\beta}$ defined by

$$(K_{\alpha,\beta}(h))(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(-A)^{-\beta}e^{(t-s)A}\,h(s)\,ds,\;\; t\in(0,T),\;\; h\in L^2(0,T;L^2(D)).$$

Let $p_\beta(t,x,y)$ be the integral kernel of the operator $(-A)^{-\beta}e^{tA}$, i.e.

$$[(-A)^{-\beta}e^{tA}g](x) = \int_D p_\beta(t,x,y)g(y)\,dy,\;\; x\in D,\;\; g\in L^2(D).$$

Then, for $(t,x) \in (0,T) \times D$ and $h \in L^2(0,T;L^2(D))$ we have

$$(K_{\alpha,\beta}(h))(t,x) = \int_{(0,T)} \int_D \frac{1}{\Gamma(\alpha)} 1_{(0,t)}(s)(t-s)^{\alpha-1} p_\beta(t-s,x,y) h(s,y)\, dyds.$$

Denote

$$\kappa_{\alpha,\beta}(t,x;s,y) = \frac{1}{\Gamma(\alpha)} 1_{(0,t)}(s)(t-s)^{\alpha-1} p_\beta(t-s,x,y),\ (t,x),(s,y) \in (0,T) \times D.$$

Then

$$\begin{aligned}
&\int_{(0,T)} \left\{ \int_D \left[\int_{(0,T)\times D} |\kappa_{\alpha,\beta}(t,x;s,y)|^2\, dyds \right]^{p/2} dx \right\}^{q/p} dt \\
&= \frac{1}{\Gamma(\alpha)^p} \int_{(0,T)} \left\{ \int_D \left[\int_{(0,t)\times D} |(t-s)^{\alpha-1} p_\beta(t-s,x,y)|^2\, dyds \right]^{p/2} dx \right\}^{q/p} dt
\end{aligned}$$

Hence, we get the following

Corollary 3.5. *Assume that $\alpha \in (0,1]$, $\beta \in [0,1)$ and $p,q \in (1,\infty)$. Then the operator $K_{\alpha,\beta}$ is γ-radonifying from $L^2(0,T;L^2(D))$ to $L^q(0,T;L^p(D))$ iff*

$$\int_{(0,T)} \left\{ \int_D \left[\int_{(0,t)\times D} |(t-s)^{\alpha-1} p_\beta(t-s,x,y)|^2\, dyds \right]^{p/2} dx \right\}^{q/p} dt < \infty. \tag{3.9}$$

If in addition A is self-adjoint with eigenvalues $\{-\lambda_j\}_j$ and the corresponding set of eigenvectors $\{e_j\}$ (which is also an ONB of $L^2(D)$) then

$$p_\beta(t,x,y) = \sum_j \lambda_j^\beta e^{-\lambda_j t} e_j(x) e_j(y),\ t > 0,\ x,y \in D$$

and therefore for $0 \le s < t$ and $x \in D$, we have

$$\int_D |(t-s)^{\alpha-1} p_\beta(t-s,x,y)|^2\, dy = (t-s)^{2\alpha-2} \sum_j \lambda_j^{2\beta} |e_j(x)|^2.$$

Hence, the operator $K_{\alpha,\beta}$ is γ-radonifying from $L^2(0,T;L^2(D))$ to $L^q(0,T;L^p(D))$ iff

$$\begin{aligned}
&\int_{(0,T)} \left\{ \int_D \left[\int_{(0,t)} (t-s)^{2\alpha-2} \sum_j \lambda_j^{2\beta} e^{-2\lambda_j(t-s)} |e_j(x)|^2\, ds \right]^{p/2} dx \right\}^{q/p} dt \\
&= \int_{(0,T)} \left\{ \int_D \left[\sum_j 2^{1-2\alpha} \lambda_j^{1+2\beta-2\alpha} \Gamma(2\alpha-1, 2\lambda_j t) |e_j(x)|^2\, ds \right]^{p/2} dx \right\}^{q/p} dt < \infty,
\end{aligned}$$

where $\Gamma(z,t)$ is the truncated Euler gamma function defined by

$$\Gamma(z,t) = \int_0^t s^{z-1} e^{-s}\, ds,\ z > 0.$$

4. Lecture 4: Martingale type 2 Banach spaces, stochastic integral in martingale type 2 Banach spaces, Burkholder-Davis inequality,

Definition 4.1. Let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, and let H be a real separable Hilbert space. By an *$(\mathcal{F}_t)$-adapted cylindrical Wiener process on $\mathcal{H}$* we understand a family $W(t)$, $t \geq 0$ of bounded linear operators from H into $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that:

(i) for all $t \geq 0$, and $\psi, \varphi \in \mathrm{H}$, $\mathbb{E}\, W(t)\psi W(t)\varphi = t\langle \psi, \varphi\rangle_{\mathrm{H}}$,
(ii) for each $\psi \in \mathrm{H}$, $W(t)\psi$, $t \geq 0$ is a real valued $\mathbb{F}$-adapted Wiener process.

Remark 4.2. Assume that E is a Banach space and $(W_t)_{t\geq 0}$ is a E-valued Wiener process defined on a filtered probability space $\mathfrak{A}$. Replacing if necessary E by its closed subspace we can assume that E is equal to the support of the law $\mathcal{L}(W_1)$ of the E-valued random variable W_1. Then, see e.g. [26], there is a unique densely and continuously imbedded into E separable Hilbert space H such that

$$\mathbb{E}\,(W_t, \psi)_{E,E^*}(W_s, \varphi)_{E,E^*} = t \wedge s\,\langle \psi, \varphi\rangle_{\mathrm{H}} \quad \text{for } t, s \geq 0,\ \psi, \varphi \in E^*,$$

where $(\cdot,\cdot)_{E,E^*}$ stands for the canonical bilinear form on $E\times E^*$, and we identify the space $\mathcal{H}^*$ with its dual H, and then we identify the dual space E^* with a properly chosen subspace of H. Thus, as E^* is dense in H, for any $t \geq 0$ the mapping

$$E^* \ni \psi \mapsto (W_t, \psi)_{E,E^*} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$$

has the unique continuous extension to H. We denote this extension by $W(t)$. Note that the juste defined family $W(t)$, $t \geq 0$ is a cylindrical Wiener process on H. The space H is called the *reproducing kernel Hilbert space*, shortly *RKHS*, or *Cameron–Martin space* of W.
Conversely, suppose that $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ is a filtered probability space, H is a real separable Hilbert space and a family $W(t)$, $t \geq 0$ of bounded linear operators from H into $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is an *$(\mathcal{F}_t)$-adapted cylindrical Wiener process on $\mathcal{H}$*. Let us then choose a separable Banach space E such that $H \hookrightarrow E$ densely and γ-radonifyingly and an ONB $\{e_k\}_{k=1}^\infty$ of H. Then one can show that there exists a set $\hat{\Omega} \in \mathcal{F}$ such that $\mathbb{P}(\hat{\Omega}) = 1$ and that the series on the RHS of the equality below

$$W_t(\omega) := \sum_{k=1}^\infty [\widetilde{w(t)e_k}](\omega)e_k,\ \omega \in \hat{\Omega},\ t \geq 0$$

is convergent in E. Here by $\widetilde{w(t)e_k}$ we mean a suitable representant of $w(t)e_k$. One can also prove that (W_t), $t \geq 0$ is an E-valued Wiener process such that the RKHK of $\mathcal{L}(W_1)$ is equal to H and that the cylindrical Wiener process corresponding to $(W_t)_{t\geq 0}$ (constructed as in the first part of this Remark) is equal to W.

Remark 4.3. Formally $W(t) = \sum_{i=1}^\infty (W(t)e_i)\, e_i$ where $\{e_i\}$ is ONB of H.

$$W(t) = \sum_{i=1}^\infty W_i(t)e_i$$

where $\{W_i(t)\}_{i=1}^\infty$ are independednt $\mathbb{R}-$valued Brownian Motion.

Remark. If (i, H, E) is an AWS then $\sum_{k=1}^{\infty} W_k(t)i(e_k)$ is convergent in $L^2(\Omega, E)$. So $W(t)$ can be seen as a $E-$valued Weiner process.

Let E be a Banach space, and let us denote by $\mathcal{M}^p(0, T; E)$ the Banach space of all $\mathbb{F}$-progressively measurable E-valued processes ξ such that

$$|\xi|_{\mathcal{M}^p(0,T;E)} \stackrel{\text{def}}{=} \left(\mathbb{E} \int_0^T |\xi(t)|_E^p \, dt\right)^{1/p} < \infty.$$

Definition 4.4. Let X be a Banach space and consider a process $\xi \colon [0, T] \times \Omega \to \gamma(H, X)$ then ξ is said to be progressively measurable iff $\forall T > 0$
$\xi : [0, T] \times \Omega \to \gamma(H, X)$ is $\mathbb{B}([0, T]) \times \mathcal{F}_T / \mathbb{B}(\gamma(H, X))$- measurable.

Definition 4.5. A process ξ is a step process if for a certain partition
$0 = t_0 < t_1 < \cdots < t_N < \infty$,

$$\xi(t) = \begin{cases} \xi(t_i), & t \in [t_i, t_{i+1}), \\ 0, & t \geq t_N. \end{cases}$$

Definition 4.6. $(M_j)_{j=0}^n$ is a martingale w.r.t. filtration $(\mathcal{G}_j)_{j=0}^n$ if M_j is $\mathcal{G}_j$-measurable, $\mathbb{E}(|M_j|) < \infty$ and $\mathbb{E}(M_{j+1}|\mathcal{G}_j) = M_j$.

We define $\mathcal{I}(\xi) \colon \Omega \to X$,

$$\mathcal{I}(\xi) = \sum_{k=0}^{N-1} \xi(t_k) \left[W(t_{k+1}) - W(t_k)\right]. \tag{4.1}$$

Introduce an auxiliary space E. $H \hookrightarrow E$ is γ-radonifying i.e. (i, H, E) is an AWS.
$\xi(t_k) = \tilde{\xi}(t_k)i$, where $\tilde{\xi} \colon \Omega \to \mathcal{L}(E, X)$.

An alernative approach is as follows

Let us fix an orthonormal basis $\{e_k\}$ of H, and let us denote by Π_n the orthogonal projection onto the space spanned by $e_1, \ldots, e_n$. Let E be a real separable Banach space, and let $\mathcal{M}_0(E)$ denote the class of all $\xi \in \mathcal{M}^2(0, T; \gamma(\mathrm{H}, E))$ such that

$$\xi(t, \omega) = \sum_{j=1}^{n} 1_{(t_j, t_{j+1}]}(t) \, [\xi_j(\omega) \circ \Pi_i]$$

for some $n, i \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_{m+1} \leq T$ and $\xi_j \in L^2(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; \gamma(\mathrm{H}, E))$. For $\xi \in \mathcal{M}_0(E)$ and $t \in [0, T]$ we put

$$\mathcal{I}_t^W(\xi) := \sum_{j=1}^{n} \sum_{k=1}^{i} \Big[(W(t_{j+1} \wedge t) - W(t_j \wedge t))e_k\Big] [\xi_j(e_k)].$$

Back to the approach presented during the lecture
In general, neither I nor $\mathcal{I}^W$ can be extended continuously to the whole $\mathcal{L}^2(0, T; R(H, E))$. This holds true for E being an M-type 2 Banach space, see e.g. [4] or [22]. Further

to make the theory of stochastic integrals applicable we need a Burkholder type inequality. It turns out that this is also true when E is an M-type 2 Banach space, see [4] and [22].

Definition 4.7. A Banach space E is of martingale type 2 if there exists a constant $L = L_2(E) > 0$ such that for every E-valued finite martingale $\{M_n\}_{n=0}^N$ the following holds:

$$\sup_n \mathbb{E}|M_n|^2 \leq L \sum_{n=0}^{N} \mathbb{E}|M_n - M_{n-1}|^2, \tag{4.2}$$

Define $M_j = \sum_{k=0}^{j-1} \xi(t_k)\,[W(t_{k+1}) - W(t_k)]$ is a $(\mathcal{F}_{t_j})$ martingale then $\mathcal{I}(\xi) = M_N$ if X is M type 2.

$$\begin{aligned}
\mathbb{E}|\mathcal{I}(\xi)|_X^2 &\leq C \sum_{k=0}^{N-1} \mathbb{E}|\xi(t_k)\,[W(t_{k+1}) - W(t_k)]\,|_X^2 \\
&= C \sum_{k=0}^{N-1} \mathbb{E}|\tilde{\xi}(t_k) \circ i\,[W(t_{k+1}) - W(t_k)]\,|_X^2 \\
&= C \int_E |\tilde{\xi}(t_k)y|_X^2 d\mu(y) \\
&= (t_{k+1} - t_k) \int_E |\tilde{\xi}(t_k)y|_X^2 d\mu^\circ(y)
\end{aligned}$$

where μ is the law of $W(t_{k+1}) - W(t_k)$ and μ° is the measure (i, H, E).

$$\begin{aligned}
\mathbb{E}|\mathcal{I}(\xi)|_X^2 &\leq C \sum_{k=0}^{N-1} (t_{k+1} - t_k)\mathbb{E}|\xi(t_k)|_{\gamma(H,X)}^2 \\
&= C\mathbb{E} \int_0^\infty |\xi(t)|_{\gamma(H,X)}^2\, dt
\end{aligned}$$

Theorem 4.8. *If $M^2(0,\infty;\gamma(H,X))$ is the space of $\mathbb{F}$-progressively measurable $\gamma(H,X)$-valued process then $\exists!$ |mathcalI: $M^2 \to L^2(\Omega; X)$ such that if ξ is a step process then $\mathcal{I}(\xi)$ can be defined by (4.1).*

Theorem 4.9. *$\int_0^t \xi(s)\, dW(s) = \mathcal{I}(1_{[0,t]}\xi)$. Moreover*

$$\mathbb{E}| \int_0^t \xi(s)\, dW(s)|_X^2 \leq C\mathbb{E} \int_0^t |\xi(s)|_{\gamma(H,X)}^2\, ds.$$

$\int_0^t \xi(s)\, dW(s)$ is a X-valued martingale wrt $(\mathcal{F}_t)_{t\geq 0}$.

Theorem 4.10. ***Burkholder-Davis-Gundy inequality*** *Let $q \in [1,\infty)$, then*

$$\mathbb{E} \sup_{0\leq s\leq t} \Big| \int_0^t \xi(s)\, dW(s)\Big|_E^q \leq C\,\mathbb{E}\Big(\int_0^t \|\xi(s)\|_{\gamma(H,X)}^2\, ds\Big)^{q/2}. \tag{4.3}$$

Back to the alternative approach
For simplicity of notation let $\mathcal{M}^p_{\text{step}}(a,b;S)$ be the space of equivalence classes (in $\mathcal{M}^p(a,b;S)$) of elements of $\tilde{\mathcal{M}}^p(a,b;S)\cap\mathcal{M}_{\text{step}}(a,b;S)$.
For $\xi\in\mathcal{M}^2_{\text{step}}(a,b;\mathcal{L}(E,X))$ (with partition $\pi=\{a=t_0<t_1<\ldots<t_n=b\}$) let $I(\xi)$ be a measurable map from Ω into X defined by

$$I(\xi):=\sum_{k=0}^{n-1}\xi(t_k)\,(w(t_{k+1})-w(t_k))\,. \tag{4.4}$$

We have

Lemma 4.11. *Assume that X is an M-type 2 Banach space and $i:H\to E$ is an AWS. Then for any $\xi\in\mathcal{M}^2_{\text{step}}(a,b;\mathcal{L}(E,X))$*

$$\begin{aligned} I(\xi) &\in L^2(\Omega,X),\\ \mathbb{E}I(\xi) &= 0 \end{aligned}$$

and

$$\mathbb{E}|I(\xi)|^2\le C_2(X)\mathbb{E}\int_a^b\|\xi(t)\|^2_{\gamma(H,X)}\,dt. \tag{4.5}$$

Remark 4.12. Neidhard proved Lemma 4.11 by using property ••) of X. An alternative approach based on using the M-type 2 property of X (i.e. (4.7)) was proposed independently in [4]. In both approaches one uses the standard properties of the conditional expectation and the following

Lemma 4.13. *If ξ is $\mathcal{L}(E,X)$ and $t>s$, then*

$$\mathbb{E}|\xi(w(t)-w(s))|^2 \;=\; (t-s)\int_X|z|^2\,d\nu_{\tilde{\imath}(\xi)}(z). \tag{4.6}$$

Lemma 4.13 is a simple consequence of part 2. of Theorem 5.8. See also Lemma III.2 in [24] and the remarks following its proof.

Digression 1 A Banach space X is an UMD space iff there exist $\beta>0$ and $p\in(1,\infty)$ such that for any X-valued martingale difference sequence $\eta=\{\eta_j\}_{j=1}^n$ and for any $\varepsilon\in\{-1,1\}^n$

$$\left\{\mathbb{E}|\sum_{i=1}^n\varepsilon_i\eta_i|_X^p\right\}^{1/p}\le\beta\left\{\mathbb{E}|\sum_{i=1}^n\eta_i|_X^p\right\}^{1/p}, \tag{4.7}$$

The smallest constant β for which (4.7) holds will be denoted by $\beta_p(X)$. This definition is p independent, see [42].
A Banach space X is of type 2 iff there exists a constant $K>0$ such that for any $x_1,\cdots,x_n\in X$ and any symmetric i.i.d. random variables $\sigma_1,\cdots,\sigma_n:\Omega\to\{-1,1\}$ the following holds

$$\left\{\mathbb{E}|\sum_{i=1}^n\sigma_ix_i|^2\right\}^{\frac12}\le K\left\{\sum_{i=1}^n|x_i|^2\right\}^{\frac12}$$

The smallest number K for which the above holds is denoted by $K_2(X)$.
On the other hand, using the Kahane inequality which asserts that for any $r\in(0,\infty)$ there

exist numbers $A_r, B_r > 0$ such that for any Banach space Z, for all $z_1, \cdots, z_n \in Z$ and for all $\sigma_1, \cdots, \sigma_n : \Omega \to \{-1, 1\}$ symmetric i.i.d. random variables

$$A_r\{\mathbb{E}|\sum_i \sigma_i z_i|^2\}^{\frac{1}{2}} \le \{\mathbb{E}|\sum_i \sigma_i z_i|^r\}^{\frac{1}{r}} \le B_r\{\mathbb{E}|\sum_i \sigma_i z_i|^2\}^{\frac{1}{2}},$$

one sees that X is of type 2 iff for any (some) $r \in (0, \infty)$ there exists $K' > 0$ such that

$$\{\mathbb{E}|\sum_i \sigma_i x_i|^r\}^{\frac{1}{r}} \le K'\{\sum_i |x_i|^2\}^{\frac{1}{2}} \tag{4.8}$$

for any $x_1, \cdots, x_n \in X$ and any $\sigma_1, \cdots, \sigma_n : \Omega \to \{-1, 1\}$ symmetric i.i.d. random variables. The smallest constant K' is denoted by $K_{2,r}(X)$.

In this context, let us note that $K_{2,r}(X) \le K_2(X)B_r$ and $K_2(X) = K_{2,2}(X)$.

The fundamental property of the mapping I is that it extends uniquely to a bounded linear map from $\mathcal{M}^2(a, b; \gamma(H, X)$ into $L^2(\Omega, X)$. This follows easily from (4.5) and the fact (proven in [35]) that $\mathcal{M}^2_{\text{step}}(a, b; \mathcal{L}(E, X))$ is dense in $\mathcal{M}^2(a, b; \gamma(H, X)$. The value of this extension at $\xi \in \mathcal{M}^2(a, b; \gamma(H, X)$ will be denoted by $\int_a^b \xi(s)\, dw(s)$.

We have the following consequence of Theorems 2.4 and 3.3 from [22], see also [35], and [6].

Theorem 4.14. *Assume that E is a martingale type 2 Banach space. Then $\mathcal{M}_0(E)$ is dense in $\mathcal{L}^2(0, T; \gamma(\mathrm{H}, E))$ and for each $t \in [0, T]$ there exists a unique extension of $\mathcal{I}_t^W$ to a linear bounded operator from $\mathcal{L}^2(0, T; \gamma(\mathrm{H}, E))$ into $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; E)$. Moreover, there exists a constant C such that for any $\xi \in \mathcal{L}^2(0, T; \gamma(\mathrm{H}, E))$ one has*

$$\mathbb{E} \sup_{s\in[0,t]} |\mathcal{I}_t^W(\sigma)|_E^2 \le C\, \mathbb{E} \int_0^T \|\sigma(s)\|^2_{\gamma(\mathrm{H},E)}\, \mathrm{d}s, \quad t \in [0, T]. \tag{4.9}$$

Furthermore, the Burkholder inequality holds also for random times, see [6, 21, 36], *i.e. for every $p \in (1, \infty)$ there exists a constant $B_p(E) > 0$ such that for each accessible stopping time $\tau > 0$ and $\gamma(\mathrm{K}, E)$-valued progressively measurable process ξ,*

$$\mathbb{E} \sup_{0\le t\le\tau} \Big| \int_0^t \xi(s)\, dW(s)\Big|_E^p \le B_p(E)\, \mathbb{E}\Big(\int_0^\tau \|\xi(t)\|^2_{\gamma(\mathrm{K},E)}\, dt\Big)^{p/2}. \tag{4.10}$$

Corollary 4.15. *Let E be a martingale type 2 Banach space and $p \in [2, \infty)$. Then there exists a constant $\hat{B}_p(E)$ depending on E such that for every $T \in (0, \infty]$ and every $L^p(0, T; E)$-valued progressively measurable process $(\Xi_s,\ s \in [0, T))$*

$$\mathbb{E}\Big| \int_0^T \Xi_s\, dW(s)\Big|^p_{L^p(0,T;E)} \le \hat{B}_p(E)\, \mathbb{E}\Big(\int_0^T \|\Xi_s\|^2_{R(\mathrm{K},L^p(0,T;E))}\, ds\Big)^{p/2}. \tag{4.11}$$

Moreover, for any $T > 0$, the above inequality (4.11) *holds true also for the space $L^p(0, T; E)$ and the integral over interval $(0, T)$ with the same constant $\hat{B}_p(E)$.*

5. Lecture 5: Stochastic convolution in Banach spaces, two approaches. Stochastic heat equation with space time white noise, space-time regularity.

5.1. **Motivating Example.** Assume that $\bar{f}$, $\bar{g} : \mathbb{R} \to \mathbb{R}$ are two measurable functions. Let $\mathcal{O} = (0, 1)$ be the unit interval in $\mathbb{R}$. Will be interested in the following stochastic initial value problem

$$du(t,x) - \Delta u(t,x)\,dt = \bar{g}(u(t,x))\,dW(t) + \bar{f}(u(t,x))\,dt, \tag{5.1}$$

$$u(t,\cdot) = 0 \text{ on } \partial\mathcal{O}, \text{ for } t > 0, \tag{5.2}$$

$$u(0,\cdot) = u_0. \tag{5.3}$$

Here $W(t)$ is a standard $H = L^2(\mathcal{O})$–cylindrical Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)$, see Definition 4.1.
For this let us fix some $q \in [2,\infty)$ and let $H = L^2(\mathcal{O})$ and $X = L^q(\mathcal{O})$. Let $A = A_q = -\Delta$ with $D(A_q) = H^{2,q}(\mathcal{O}) \cap H_0^{1,q}(\mathcal{O})$. We would like to study the space-time regularity of the solution u of the above problem. For this purpose let us assume that u is a solution, in a sense which will written below and let us define an $\mathcal{L}$-valued process η by formula

$$\eta : [0,\infty) \times \Omega \ni (t,\omega) \mapsto \{H \ni h \mapsto \bar{g}(u(t,\cdot,\omega))h(\cdot) \in H\}.$$

Since $\bar{g}$ is a bounded function, we infer that η is well defined. Thus u solves the following problem,

$$du(t,x) - \Delta u(t,x)\,dt = \bar{g}(u(t,x))\,dW(t) + \bar{f}(u(t,x))\,dt, \tag{5.4}$$

together with the boundary and initial conditions (5.2) and (5.3).

Definition 5.1. An $\mathbb{F}$-adapted and X–valued continuous process $u(t)$, $t \in [0,T]$, is called a mild solution to problem (5.4), (5.2) and (5.3), iff there exists a number $p \in [2,\infty)$ and $\gamma > 0$ such that

(i) $u \in \mathcal{M}^p(0,T; D(A^\gamma)) \cap L^p(\Omega; C([0,T]; D(A^\delta)))$;
(ii) for each $t \in [0,T]$, the following equality holds a.s.,

$$u(t) = \int_0^t e^{-(t-s)A}\,[\eta(s)]\,dw(s) + e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\left[\bar{f}(u(s))\right]ds. \tag{5.5}$$

We shall prove more general results, see Theorem 5.8 and Corollary 5.11, from which we will be able to deduce the following regularity of the stochastic convolution process in (5.5).

Theorem 5.2. *Suppose that $T > 0$, $p > 4$, $q \geq 2$ and that the stochastic process η belongs to $\mathcal{M}^p(0,T;\mathcal{L}(H))$. Let*

$$\mathrm{v}_q(t) = \int_0^t e^{-(t-s)A_q}\eta(s)\,dw(s), \; t \in [0,T]. \tag{5.6}$$

Supposing that

$$\beta + \delta + \frac{1}{p} < \frac{1}{4}, \tag{5.7}$$

(i) *there exists a modification* $\tilde{\mathrm{v}}_q(t)$, $t \in [0,T]$ *of the process* $\mathrm{v}_q(t)$ *such that, for some constant* C_T *independent of* η *,*

$$\tilde{\mathrm{v}}_q(\cdot,\omega) \quad \in \quad C^\beta([0,T];H_0^{2\delta,q}(\mathcal{O})), \quad a.s.\ in\ \omega\in\Omega, \tag{5.8}$$

$$\mathbb{E} \parallel \tilde{\mathrm{v}}_q \parallel^p_{C^\beta([0,T];H_0^{2\delta,q}(\mathcal{O}))} \quad \le \quad C_T\mathbb{E}\int_0^T |\eta(s)|^p_{\mathcal{L}(H)}\,ds. \tag{5.9}$$

(iii) *if* $q_2 > q_1$ *then the processes* $\tilde{\mathrm{v}}_{q_2}$ *and* $\tilde{\mathrm{v}}_{q_1}$ *are equivalent.*
(iii) *If* $\kappa \ge 0$ *satisfies*

$$\beta + \frac{\kappa}{2} + \frac{1}{p} < \frac{1}{4}, \tag{5.10}$$

then there exists a modification $\tilde{\mathrm{v}}(t)$ *of the process* $\mathrm{v}_q(t)$ *which satisfies*

$$\tilde{\mathrm{v}}(\cdot,\omega) \quad \in \quad C^\beta([0,T];C_0^\kappa(\mathcal{O})), \quad a.s.\ in\ \omega\in\Omega, \tag{5.11}$$

$$\mathbb{E} \parallel \tilde{\mathrm{v}} \parallel^p_{C^\beta([0,T];C_0^\kappa(\mathcal{O}))} \quad \le \quad C_T\mathbb{E}\int_0^T |\eta(s)|^p_{\mathcal{L}(H)}\,ds. \tag{5.12}$$

Remark 5.3. One can take $p = \infty$. Then the conditions (5.7) and (5.10) become very simple: $\beta + \delta + \frac{1}{p} < \frac{1}{4}$ and $\beta + \frac{\kappa}{2} < \frac{1}{4}$.

In order to deduce Theorem 5.2 from other results as claimed above we need the following auxiliary claim.

Proposition 5.4. *The operator* $A_q^{-\theta}$ *belongs to* $\gamma(H,X)$ *iff* $\theta > \frac{1}{4}$.

Proof of Proposition 5.4. It is quite obvious that $A_q^{-\theta}$ belongs to $\gamma(H,X)$ iff $A_q^{-\theta}$ belongs to $\gamma(D(A_2^{\frac{1}{2}}), D(A_q^{\frac{1}{2}})$. Since $A_q^{-\theta}$ is an isomorphism between $D(A_q^{\frac{1}{2}-\theta})$ and $D(A_q^{\frac{1}{2}})$, we infer $A_q^{-\theta}$ belongs to $\gamma(H,X)$ iff the natural embedding from $D(A_2^{\frac{1}{2}})$ to $D(A_q^{\frac{1}{2}-\theta})$, i..e from $H_0^{1,2}((0,1))$ to $H_0^{1-2\theta,q}((0,1))$, is γ-radonifying. The latter holds iff $1-(1-2\theta) > \frac{1}{2}$, i.e. $\theta > \frac{1}{4}$. The proof is complete. □

Thus we deduce the following.

Corollary 5.5. *If* $\theta > \frac{1}{4}$*, then for all* $(t,\omega) \in [0,T]\times\Omega$*,* $A_q^{-\theta}\eta(t,\omega) \in \gamma(H,X)$ *and moreover, there exists* $C > 0$ *such that*

$$\|A_q^{-\theta}\eta(t,\omega)\|_{\gamma(H,X)} \le C, \quad (t,\omega)\in[0,T]\times\Omega. \tag{5.13}$$

5.2. **Abstract theory.** Let us list assumptions we assume throughout this section (unless otherwise specified).

(H1) X is and martingale type 2 Banach space.
(H2) A is a densely defined closed, positive operator in X, in particular $(\lambda+A)^{-1} \in \mathcal{L}(X)$ for $\lambda > 0$ and (for some $M > 0$)

$$\|(A+\lambda)^{-1}\| \le \frac{M}{\lambda}, \ \lambda > 0.$$

Moreover, $-A$ is a generator of an analytic semigroup $\left\{e^{-tA}\right\}_{t\ge0}$ on X.
(H3) A is one-to-one (but we don't assume that A^{-1} is bounded).

(H4) H is a separable Hilbert space. The H-canonical cylindrical Wiener process is denoted by $w(t)$, $t \geq 0$.

Assume that ξ is an $\gamma(H, X)$-valued progressively measurable process. Detailed assumptions on ξ will be formulated later. We will study now regularity of the solution u to the following equation

$$du(t) + Au(t)\,dt = \xi(t)\,dW(t), \tag{5.14}$$

$$u(0) = 0 \tag{5.15}$$

Definition 5.6. An $\mathbb{F}$-adapted and X–valued continuous process $u(t)$, $t \in [0, T]$, is called a mild solution to problem (5.14)-(5.15), iff there exists a number $p \in [2, \infty)$ and $\gamma > 0$ such that

(I) $u \in \mathcal{M}^p(0, T; D(A^\gamma)) \cap L^p(\Omega; C([0, T]; X))$;

(II) for each $t \in [0, T]$, the following equality holds a.s.,

$$u(t) = \int_0^t e^{-(t-s)A}\xi(s)\,dw(s). \tag{5.16}$$

For fixed $p \in (1, \infty)$, $T \in (0, \infty)$ let B_T be a linear (unbounded) operator in $L^p(0, T; X)$ defined by

$$B_T u = u', \ u \in D(B_T), \tag{5.17}$$

$$D(B_T) = {}_0\mathrm{H}^{1,p}(0, T; X), \tag{5.18}$$

where $\mathrm{H}^{1,p}(0, T; X)$ is the Banach space of functions $u \in L^p(0, T; X)$ whose weak derivative u' belongs to $L^p(0, T; X)$ and, in somehow unorthodox notation,

$${}_0\mathrm{H}^{1,p}(0, T; X) = \left\{u \in \mathrm{H}^{1,p}(0, T; X) : u(0) = 0\right\}. \tag{5.19}$$

The Sobolev imbedding theorem implies that $H^{1,p}(0, T; X)$ can be continuously imbedded into $C([0, T]; X)$, the space of continuous functions from the closed interval $[0, T]$ with values in X, and in consequence, the space ${}_0\mathrm{H}^{1,p}(0, T; X)$ from (5.19) is well defined.

If $\alpha \in (0, 1)$ then the Sobolev space $H^{\alpha,p}(0, T; X)$ is defined by

$$H^{\alpha,p}(0, T; X) = [L^p(0, T; X), H^{1,p}(0, T; X)]_\alpha, \tag{5.20}$$

where $[\cdot, \cdot]_\alpha$ denotes the complex interpolation space of order α, see [40].

Finally we note that the original norm on ${}_0\mathrm{H}^{1,p}(0, T; X)$ (i.e. the one inherited from $H^{1,p}(0, T; X)$) is equivalent to the following one

$$||u||^p = \int_0^T |u'(t)|^p\,dt. \tag{5.21}$$

It is known that the operator B_T generates a C_0-semigroup on the Banach space $L^p(0, T; X) =: Y_T$ and this semigroup, denoted by $\{S(t)\}_{t\geq 0}$, acts through the formula

$$[S(t)u](r) = \begin{cases} u(t-r), & \text{if } 0 \leq t \leq r, \\ 0, & \text{if } 0 \leq r < t, \end{cases} \tag{5.22}$$

for $r \in [0,T]$ and $u \in L^p(0,T;X)$. Moreover, see [23], if X is an UMD Banach space then B_T is a positive operator and for some constant $C = C_T(X) > 0$

$$\|B_T^{is}\| \leq C(1+s^2)e^{\frac{\pi}{2}|s|}, \;\; s \in \mathbb{R}. \tag{5.23}$$

Therefore it follows from [40], see [5],

$$ {}_0\mathrm{H}^{\alpha,p}(0,T;X) \;\; = \;\; \mathrm{R}(B_T^{-\alpha}), \;\; \frac{1}{p} < \alpha < 1, \tag{5.24}$$

where $B_T^{-\alpha}$, the fractional power of B_T of order $-\alpha$, $R(B_T^{-\alpha})$ denotes the range of this operator and

$$ {}_0\mathrm{H}^{\alpha,p}(0,T;X) \;\; := \;\; \left[L^p(0,T;X); {}_0\mathrm{H}^{1,p}(0,T;X)\right]_\alpha .$$

Let us recall that for $z \in \mathbb{C}$ such that $1 > \mathrm{Re}\, z > 0$ and for $f \in Y$,

$$B_T^{-z} f = \frac{1}{\Gamma(1-z)\Gamma(z)} \int_0^\infty \lambda^{-z}(B_T + \lambda I)^{-1} f \, d\lambda. \tag{5.25}$$

Let us define a linear operator $\mathcal{A}_T$ in $L^p(0,T;X)$ by the formula

$$\begin{aligned} D(\mathcal{A}_T) \;\; &= \;\; \{u : [0,T] \to X \text{ s.th. } Au \in L^p(0,T;X)\}, \\ \mathcal{A}_T u \;\; &:= \;\; \{[0,T] \ni t \mapsto Au(t) \in X\}. \end{aligned} \tag{5.26}$$

Let also

$$\Lambda_T \;\; := \;\; B_T + \mathcal{A}_T, \tag{5.27}$$

$$D(\Lambda_T) \;\; := \;\; D(B_T) \cap D(\mathcal{A}_T). \tag{5.28}$$

Λ_T is a closable nonnegative operator. Completion of its domain $D(\Lambda_T)$ with respect to a norm

$$\|u\| = \left\{ \int_0^T |u'(s)|^p \, ds + \int_0^T |Au(s)|^p \, ds \right\}^{\frac{1}{p}} \tag{5.29}$$

is a Banach space. It will be frequently denoted by ${}_0\mathrm{H}^{1,p}(0,T;X,A)$.
We begin the exposition of our results by presenting an explicit formula for the fractional power of the operator Λ_T.

Proposition 5.7. *Suppose that X is a Banach space and that the assumption* **(H2)** *is satisfied. Then, for* $0 < \alpha < 1$, $f \in L^p(0,T;X)$ *and* $t \in (0,T)$*, we have*

$$\left(\Lambda_T^{-\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-(t-s)A} f(s)\, ds. \tag{5.30}$$

Note 1It is an easy application of the Young inequality to verify that, for $0 < \alpha < 1$*, the RHS of* (5.30) *defines a linear bounded operator in* $L^p(0,T;X)$.
Proof of Proposition 5.7. Take $\lambda > 0$. Then $\lambda I + \Lambda_T$ is invertible (with bounded inverse) and moreover is given by the following explicit formula, see [5]

$$\left[(\lambda I + \Lambda_T)^{-1} f\right](r) = \int_0^r e^{-(r-s)(\lambda I + A)} f(s)\, ds,$$

for $f \in L^p(0,T;X)$ and $r \in [0,T]$. Using then (5.25) with Λ_T in place of B_T and applying the Fubini Theorem, we have

$$\begin{aligned} \left(\Lambda_T^{-\alpha} f\right)(r) &= \int_0^r e^{-(r-s)A} f(s) \left[\int_0^\infty \lambda^{-\alpha} e^{-\lambda(r-s)}\, d\lambda\right] ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^r e^{-(r-s)A} f(s)(r-s)^{\alpha-1}\, ds \end{aligned}$$

what concludes the proof of (5.30).

□

Before we formulate the next result let us remind that according to Definition 5.6, the process x defined below in formula (5.33) is a mild solution to equation (5.14).

Theorem 5.8. *Suppose that the conditions* **(H1)–(H4)** *hold. Assume that* $p \geq 2$, $T \in (0,\infty)$ *and that* $\alpha, \vartheta \in [0, \frac{1}{2})$ *satisfy*

$$\alpha + \vartheta < \frac{1}{2}. \tag{5.31}$$

Assume also that the stochastic process ξ *is such that*

$$A^{-\vartheta}\xi \in \mathcal{M}^p(0,T;\gamma(H,X)). \tag{5.32}$$

For $t \in [0,T]$ *we set*

$$x(t) := \int_0^t e^{-A(t-s)}\xi(s)\, dw(s). \tag{5.33}$$

Then, there exists a modification $\tilde{x}(t)$, $t \in [0,T]$, *of the process* $x(t)$, *i.e. a process satisfying*

$$x(t) = \tilde{x}(t), \text{ a.s., for each } t \in [0,T], \tag{5.34}$$

such that, for a constant $C = C(\alpha, p, X, A)$ *independent of* ξ *and* T,

$$\tilde{x} \in D(\Lambda_T^\alpha)(= R(\Lambda_T^{-\alpha})), \tag{5.35}$$

$$\mathbb{E}\|\tilde{x}\|^p_{D(\Lambda_T^\alpha)} \leq CT^{(\frac{1}{2}-(\alpha+\vartheta))p}\mathbb{E}\int_0^T \|A^{-\vartheta}\xi(s)\|^p_{\gamma(H,X)}\, ds. \tag{5.36}$$

Note 2 The condition (5.32) *can be expressed in a, more or less, equivalent form as*

$$\xi \in \mathcal{M}^p(0,T;M(H,D(A^{-\vartheta}))), \tag{5.37}$$

where $D(A^{-\vartheta})$ *is a certain extrapolation space, see* [4].

The idea of the proof is taken from [18] (where only the Hilbert space case is considered) and is an extension of the modification used by the author in [5].

Proof of Theorem 5.8. Take and fix α satisfying (5.31). Define a process $y(t)$, $t \in [0,T]$, by

$$y(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-(t-s)A}\xi(s)\, dw(s). \tag{5.38}$$

Then the Burkholder inequality (4.3) yields that, for $t \in [0,T]$,

$$\mathbb{E}|y(t)|^p_X \leq c_p \mathbb{E}\left\{\int_0^t (t-s)^{-2\alpha}\|e^{-(t-s)A}\xi(s)\|^2_{\gamma(H,X)}\, ds\right\}^{\frac{p}{2}}. \tag{5.39}$$

Since $\|e^{-(t-s)A}\xi(s)\|_{\gamma(H,X)} \le C(t-s)^{-\vartheta}\|A^{-\vartheta}\xi(s)\|_{\gamma(H,X)}$ we get

$$\mathbb{E}\int_0^T |y(t)|^p \le c_p \mathbb{E}\int_0^T \left\{\int_0^t (t-s)^{-2(\alpha+\vartheta)}\|A^{-\vartheta}\xi(s)\|^2_{\gamma(H,X)}\,ds\right\}^{\frac{p}{2}}\,dt. \tag{5.40}$$

Since

$$\int_0^T \left\{\int_0^t (t-s)^{-2(\alpha+\vartheta)}\|A^{-\vartheta}\xi(s)\|^2_{\gamma(H,X)}\,ds\right\}^{\frac{p}{2}}\,dt = |h_1 * h_2|^{\frac{p}{2}}_{L^{\frac{p}{2}}(0,T)},$$

where $h_1(s) = 1_{(0,T]}(s)s^{-2(\alpha+\vartheta)}$, $h_2(s) = 1_{(0,T]}(s)\|A^{-\vartheta}\xi(s)\|^2_{\gamma(H,X)}$, by applying pathwise the Young Inequality we get

$$\begin{aligned}\int_0^T \left\{\int_0^t (t-s)^{-2(\alpha+\vartheta)}\|A^{-\vartheta}\xi(s)\|^2_{\gamma(H,X)}\,ds\right\}^{\frac{p}{2}}\,dt \\ \le\ (1-2(\alpha+\vartheta))^{-\frac{p}{2}}T^{(\frac{1}{2}-(\alpha+\vartheta))p}\int_0^T \|A^{-\vartheta}\xi(s)\|^p_{\gamma(H,X)}\,ds.\end{aligned} \tag{5.41}$$

Indeed, we have

$$\begin{aligned}|h_1|_{L^1(0,T)} &= \int_0^T s^{-2(\alpha+\vartheta)}\,ds = (1-2(\alpha+\vartheta))^{-1}T^{1-2(\alpha+\vartheta)},\\ |h_2|_{L^{\frac{p}{2}}(0,T)} &= \left(\int_0^T \|A^{-\vartheta}\xi(s)\|^{2\frac{p}{2}}_{\gamma(H,X)}\,ds\right)^{\frac{2}{p}} = |A^{-\vartheta}\xi|^2_{L^p(0,T;\gamma(H,X))}.\end{aligned}$$

Taking expectation of (5.41) and using (5.40) we obtain

$$\mathbb{E}\int_0^T |y(t)|^p\,dt \le C_p(T)\mathbb{E}\int_0^T \|A^{-\vartheta}\xi(s)\|^p_{\gamma(H,X)}\,ds, \tag{5.42}$$

where $C_p(T) = c_p(1-2(\alpha+\vartheta))^{-\frac{p}{2}}T^{(\frac{1}{2}-(\alpha+\vartheta))p}$. In particular, $y \in L^p(0,T;X)$ a.s. Now we are ready to define the process $\tilde{x}(t)$. We define it pathwise, for fixed $\omega \in \Omega$ such that $y(\cdot,\omega) \in L^p(0,T;X)$, by

$$\tilde{x}(\cdot,\omega) = \Lambda_T^{-\alpha}\left(y(\cdot,\omega)\right). \tag{5.43}$$

Then, $\tilde{x} \in D(\Lambda_T^{\alpha})$ a.s. and by (5.42), the inequality (5.36) holds. It remains to verify the condition (5.34). Before we proceed any further let us explain the idea of the proof. If in (5.33) and (5.38) the Itô integral were replaced by the Bochner one, i.e. if $\xi(s)$ were a (deterministic) element of $L^p(0,T;X)$ and $x(t) = \int_0^t e^{-(t-s)A}\xi(s)\,ds$, and if finally $y(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-(t-s)A}\xi(s)\,ds$ then, in view of Proposition (5.7) we would have $x = \Lambda_T^{-1}\xi$, $y = \Lambda_T^{-(1-\alpha)}\xi$. Then the equality $x = \Lambda_T^{-\alpha}y$ would follow since $\Lambda_T^{-1} = \Lambda_T^{-\alpha}\Lambda_T^{-(1-\alpha)}$, see [38].

For proving (5.34) let us fix $t \in [0,T]$. Then, in view of the definition (5.38) of $y(t)$ by virtue of Proposition 5.7, we have, a.s. in $\omega \in \Omega$

$$\tilde{x}(t,\omega) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-(t-s)A}y(s,\omega)\,ds.$$

Hence, the stochastic Fubini Theorem, see [4], Proposition 3.3(v), yields

$$\begin{aligned}\tilde{x}(t) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-(t-s)A}\int_0^s (s-\sigma)^{-\alpha}e^{-(s-\sigma)A}\xi(\sigma)\,dw(\sigma)\,ds\\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \left[\int_\sigma^t (t-s)^{\alpha-1}e^{-(t-s)A}e^{-(s-\sigma)A}(s-\sigma)^{-\alpha}\,ds\right]\xi(\sigma)\,dw(\sigma).\end{aligned}$$

Using the well known formula about Euler Beta function, $\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_\sigma^t (t-s)^{\alpha-1}(s-\sigma)^{-\alpha}\,ds = 1$, we finally arrive at

$$\tilde{x}(t) = \int_0^t e^{-(t-\sigma)A}\xi(\sigma)\,dw(\sigma)$$

what, in particular, yields (5.34).

■

Next we will state the following Lemma which is a direct (!) generalization of its Hilbertian version from [18, Lemma 2].

Lemma 5.9. *Assume that X is a Banach space and $-A$ is a generator of an analytic semigroup*[2] *e^{-tA}, $t \geq 0$ on X. Assume that $T \in (0,\infty)$, and the positive numbers α, β, δ satisfy*

$$0 < \beta + \delta < \alpha - \frac{1}{p} + \gamma. \tag{5.44}$$

Then, the operator $\Lambda_T^{-\alpha}$, see (5.30), maps boundedly $L^p(0,T;D(A^\gamma))$ into $C^\beta([0,T];D(A^\delta))$.

We also have the following apparently standard result.

Lemma 5.10. *If $0 \leq \beta < \alpha \leq 1$ then*

$$D(\Lambda_T^\alpha) \hookrightarrow L^p(0,T;D(A^\beta)). \tag{5.45}$$

Proof. It suffices to combine the Young inequality with Proposition 5.7 while remembering that $\|A^\beta e^{-tA}\|_{\mathcal{L}(X)} \leq C_T t^{-\beta}$, for $t \in [0,T]$, for some $C_T > 0$.

□

The above Lemmas in conjunction with Theorem 5.8 yield the following

Corollary 5.11.(i) *If nonnegative numbers β, δ and ϑ satisfy the following*

$$\beta + \delta + \frac{1}{p} < \frac{1}{2} - \vartheta, \tag{5.46}$$

then any process ξ satisfying the condition (5.32) possesses a modification $\tilde{x}(t)$, $t \in [0,T]$, that satisfies the following conditions

$$\tilde{x}(\cdot,\omega) \in C^\beta([0,T];D(A^\delta)), \quad a.s.\ in\ \omega \in \Omega, \tag{5.47}$$

$$\mathbb{E}\|\tilde{x}\|^p_{C^\beta([0,T];D(A^\delta))} \leq C_T\mathbb{E}\int_0^T \|A^{-\vartheta}\xi(s)\|^p_{\gamma(H,X)}\,ds, \tag{5.48}$$

for some constant $C_T = C_T(X,A,\alpha,p) > 0$ independent of ξ.

[2] Then there exists $\beta_0 \in \mathbb{R}$ such that the fractional powers $(\beta I + A)^{-\alpha}$, for $0 < \alpha < 1$ and $\beta \geq \beta_0$ do exist. The spaces $D(\beta I + A)^\alpha := R(\beta I + A)^{-\alpha}$ are β independent with mutually equivalent norms and one sets $D(A^\alpha) := D(\beta I + A)^\alpha$.

(ii) *If the condition* (5.31) *is satisfied then any process* ξ *satisfying the condition* (5.32) *possesses a modification* $\tilde{x}(t)$, $t \in [0,T]$, *that satisfies the following conditions*

$$\tilde{x} \in \mathcal{M}^p(0,T;D(A^\alpha)), \tag{5.49}$$

$$\mathbb{E}\int_0^T |\tilde{x}(s)|^p_{D(A^\alpha)} \leq CT^{(\frac{1}{2}-(\alpha+\vartheta))p}\mathbb{E}\int_0^T \|A^{-\vartheta}\xi(s)\|^p_{\gamma(H,X)}\,ds \tag{5.50}$$

for some constant $C = C(X, A, \alpha, p) > 0$ *independent of* ξ *and* T.

Proof of Corollary 5.11. The part (ii) follows easily from Theorem 5.8 and Lemma 5.10. Concerning part (i) let us first note that due to (5.46) there exists a number α such that

$$\beta + \delta + \frac{1}{p} < \alpha < \frac{1}{2} - \vartheta.$$

Let us choose such an α. Then the condition (5.31) is satisfied and hence, by Theorem 5.8, there exists a modification $\tilde{x}(t)$, $t \in [0,T]$ of the process $x(t)$ such that $\tilde{x}(\cdot,\omega) \in D(\Lambda_T^\alpha)$ a.s. in $\omega \in \Omega$. Since, $\beta + \delta < \alpha - \frac{1}{p}$, in view of Lemma 5.9, $D(\Lambda_T^\alpha) = R(D(\Lambda_T^{-\alpha}))$ is contained in $C^\beta([0,T];D(A^\delta))$. The proof is complete. □

Note 3 Assume that two pairs of numbers (β_i,δ_i), $i = 1,2$ *both satisfy the condition* (5.46). *Denote by* $\tilde{x}_i$, $i = 1,2$ *the corresponding process given by Corollary 5.11 (i). Then, with* $\delta = \delta_1 \wedge \delta_2$, *both* $\tilde{x}_i$, $i = 1,2$ *are continuous* $D(A^\delta)$*-valued processes. Furthermore, they are equivalent processes. Indeed, let us choose* α *such that*

$$(\beta_1 + \delta_1) \vee (\beta_2 + \delta_2) + \frac{1}{p} < \alpha < \frac{1}{2} - \vartheta.$$

Let $y_\alpha(t)$, $t \in [0,T]$ *be the process given by* (5.38). *Our claim follows from the construction* (5.43), *i.e.*

$$\tilde{x}_i(\cdot,\omega) = \Lambda_T^{-\alpha}(y_\alpha(\cdot,\omega)).$$

6. Lecture 7: Applications to stochastic Navier-Stokes Equations, weak and strong solutions, Skorokhod-Jakubowski and Prokhorov Theorems

The aim of this lecture is to study the stochastic Navier-Stokes equations with multiplicative noise, that is the equations of motion of a viscous incompressible fluid with two forcing terms, one is deterministic and the other one is random in the whole euclidean space $\mathbb{R}^d$, $d = 2, 3$. The equations are

$$\begin{cases} \partial_t \mathrm{v} + [-\nu\Delta \mathrm{v} + (\mathrm{v}\cdot\nabla)\mathrm{v} + \nabla p]\, dt = G(\mathrm{v})dw + f\, dt \\ \operatorname{div} \mathrm{v} = 0 \end{cases} \tag{6.1}$$

where the unknowns are the velocity $\mathrm{v} = \mathrm{v}(t, x)$ and the pressure $p = p(t, x)$. By $\nu > 0$ we denote the viscosity coefficient and in our model the stochastic force can depend on the velocity itself. The above equations are associated with an initial condition

$$\mathrm{v}(0, x) = \mathrm{v}_0(x) \tag{6.2}$$

where v_0 is a divergence free square integrable vector field on $\mathbb{R}^d$.

6.1. **Mathematical framework.** For $1 \le p < \infty$ let $L^p = [L^p(\mathbb{R}^d)]^d$ with norm

$$\|\mathrm{v}\|_{L^p} = \left(\sum_{k=1}^{d} \|\mathrm{v}^k\|^p_{L^p(\mathbb{R}^d)}\right)^{\frac{1}{p}}, \quad \mathrm{v} = (\mathrm{v}^1, \dots, \mathrm{v}^d).$$

Set $J^s = (I - \Delta)^{\frac{s}{2}}$. We define the generalized Sobolev spaces of divergence free vector distributions as

$$H^{s,p} = \{u \in [\mathcal{S}'(\mathbb{R}^d)]^d : \|J^s u\|_{L^p} < \infty\}, \tag{6.3}$$

$$H^{s,p}_{\mathrm{sol}} = \{u \in H^{s,p} : \operatorname{div} u = 0\} \tag{6.4}$$

for $s \in \mathbb{R}$ and $1 \le p \le \infty$. The divergence has to be understood in the weak sense. We have, see [3], that J^σ is an isomorphism between $H^{s,p}$ and $H^{s-\sigma,p}$. Moreover $H^{s_2,p} \subset H^{s_1,p}$ when $s_1 < s_2$ and the dual space of $H^{s,p}$ is the space H^{-s}_q with $1 < q \le \infty$: $\frac{1}{p} + \frac{1}{q} = 1$. we denote by $\langle\cdot,\cdot\rangle$ the $H^{s,p} - H^{-s,q}$ duality bracket:

$$\langle u, \mathrm{v}\rangle = \sum_{k=1}^{d} \int_{\mathbb{R}^d} (J^s u^k)(x)\, (J^{-s}\mathrm{v}^k)(x) dx.$$

In particular, for the Hilbert case $p = 2$ we set $\mathrm{H} = H^{0,2}_{\mathrm{sol}}$ and, for $s \neq 0$, $\mathrm{H}^s = H^{s,2}_{\mathrm{sol}}$; that is

$$\mathrm{H} = \{\mathrm{v} \in [L^2(\mathbb{R}^d)]^d : \operatorname{div} \mathrm{v} = 0\}$$

with scalar product inherited from $[L^2(\mathbb{R}^d)]^d$.

We recall the Sobolev embedding theorem, see, e.g., [3, Th. 6.5.1]. If $1 < q < p < \infty$ with

$$\frac{1}{p} = \frac{1}{q} - \frac{r-s}{d}$$

then the following inclusion holds

$$H^{r,q} \subset H^{s,p}$$

and there exists a constant C (depending on $r - s, p, q, d$) such that

$$\|v\|_{H^{s,p}} \le C\|v\|_{H^{r,q}} \quad \text{for all } v \in [\mathcal{S}'(\mathbb{R}^d)]^d.$$

Let $A = -\Pi\Delta$, where Π is the projector onto the space of divergence free vector fields. Then A is a linear unbounded operator in $H^{s,p}$ as well as in $H^{s,p}_{\text{sol}}$ ($s \in \mathbb{R}$, $1 \le p < \infty$), which generates a contractive and analytic C_0-semigroup $\{e^{-tA}\}_{t\ge 0}$. Moreover, for $t > 0$ the operator e^{-tA} is bounded from $H^{s,p}_{\text{sol}}$ into $H^{s',p}_{\text{sol}}$ with $s' > s$ and there exists a constant M (depending on $s' - s$ and p) such that, see Lemma 1.2 in the Kato-Ponce paper [28],

$$\|e^{-tA}v\|_{H^{s',p}_{\text{sol}}} \le M(1 + t^{-(s'-s)/2})\|v\|_{H^{s,p}_{\text{sol}}} \tag{6.5}$$

i.e.

$$\|e^{-tA}\|_{\mathcal{L}(H^{s,p}_{\text{sol}};H^{s',p}_{\text{sol}})} \le M(1 + t^{-(s'-s)/2}). \tag{6.6}$$

We set

$$\|\nabla v\|^2_{L^2} = \sum_{k=1}^{d} \|\nabla v^k\|^2_{L^2}, \quad v \in H^1.$$

We have $A : H^1 \to H^{-1}$ and

$$\langle Av, v\rangle = \|\nabla v\|^2_{L^2}, \quad v \in H^1.$$

Moreover

$$\|v\|^2_{H^1} = \|v\|^2_{L^2} + \|\nabla v\|^2_{L^2} \tag{6.7}$$

We define the bilinear operator $B : H^1 \times H^1 \to H^{-1}$ as

$$\langle B(u, v), z\rangle = \int_{\mathbb{R}^d} (u(x) \cdot \nabla)v(x) \ \cdot z(x)\, dx.$$

The form B is bounded, since by Hölder and Sobolev inequalities

$$|\langle B(u, v), z\rangle| \le \|u\|_{L^4}\|\nabla v\|_{L^2}\|v\|_{L^4} \ \le C\|u\|_{H^1}\|v\|_{H^1}\|z\|_{H^1}. \tag{6.8}$$

Moreover, see e.g. [39],

$$\langle B(u, v), z\rangle = -\langle B(u, z), v\rangle, \qquad \langle B(u, v), v\rangle = 0. \tag{6.9}$$

Using (6.8)-(6.9) and the fact that H^1 is dense in L^4, B can be extended to a bounded bilinear operator from $L^4 \times L^4$ to H^{-1} and

$$\|B(u, v)\|_{H^{-1}} \le \|u\|_{L^4}\|v\|_{L^4}. \tag{6.10}$$

We shall need an estimate of $B(u, v)$ in bigger spaces.

Lemma 6.1. *Let $d = 2$ and $g \in (0, 1)$. Then there exists a constant C_g such that for all* $u \in H^{1-g}$, $v \in H^{\frac{1-g}{2}}$,

$$\|B(u, v)\|_{H^{-1-g}} + \|B(v, u)\|_{H^{-1-g}} \le C_g\|u\|^{\frac{1-g}{2}}_{H^{-g}}\|u\|^{\frac{1+g}{2}}_{H^{1-g}}\|v\|_{H^{\frac{1-g}{2}}}. \tag{6.11}$$

Finally, we define the noise forcing term. Given a real separable Hilbert space Y we consider a Y-cylindrical Wiener process w defined on a stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathbb{F}_t\}_{t\geq 0}$ is a right continuous filtration, see e.g. [18]. This means that

$$w(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j, \quad t \geq 0,$$

where $\{e_j\}_{j\in\mathbb{N}}$ is a complete orthonormal system in Y and $\{\beta_j\}_{j\in\mathbb{N}}$ is a sequence of indipendent identically distributed standard Wiener processes defined on $(\Omega, \mathbb{F}, \mathbb{P})$.

6.2. **Assumptions and formulations of the results.** In one of the previous sections we recalled the definition and basic properties of γ-radonifying operators. For the covariance of the noise we make the following assumptions:

(G1) $\exists\, g \in (0,1)$ such that the mapping $G : H \to \gamma(Y; H^{-g})$ is well defined, continuous and

$$\sup_{\mathrm{v}\in H} \|G(\mathrm{v})\|_{\gamma(Y;H^{-g})} =: K_{g,2} < \infty$$

(G2) $\exists\, g \in (0,1)$ such that the mapping $G : H \to \gamma(Y; H^{-g,4}_{\mathrm{sol}})$ is well defined, continuous and

$$\sup_{\mathrm{v}\in H} \|G(\mathrm{v})\|_{\gamma(Y;H^{-g,4}_{\mathrm{sol}})} =: K_{g,4} < \infty$$

(G3) If assumption **(G1)** holds, then G extends to a Lipschitz continuous map $G : H^{-g} \to \gamma(Y; H^{-g})$, i.e.

$$\exists\, L_g > 0 : \ \|G(\mathrm{v}_1) - G(\mathrm{v}_2)\|_{\gamma(Y;H^{-g})} \leq L_g \|\mathrm{v}_1 - \mathrm{v}_2\|_{H^{-g}} \qquad \forall \mathrm{v}_1, \mathrm{v}_2 \in H^{-g}.$$

Projecting the first equation of (6.1) onto the space of divergence free vector fields, we get rid of the pressure term and we can write the stochastic Navier-Stokes equations (6.1) in abstract form as

$$\begin{cases} d\mathrm{v}(t) + [A\mathrm{v}(t) + B(\mathrm{v}(t), \mathrm{v}(t))]dt = G(\mathrm{v}(t))\, dw(t) + f(t)\, dt, & t \in (0,T] \\ \mathrm{v}(0) = \mathrm{v}_0 \end{cases} \tag{6.12}$$

We assume from now on that $\mathrm{v}_0 \in H$ and $f \in L^p(0,T; H^{-1})$ for some $p > 2$, see Assumption **A.2** and Theorem 5.1 in [11]. Now, we denote by $C([0,T]; H_{\mathrm{w}})$ the space of H-valued weakly continuous functions with the topology of uniform weak convergence on $[0,T]$; in particular $\mathrm{v}_n \to \mathrm{v}$ in $C([0,T]; H_{\mathrm{w}})$ means

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} |(\mathrm{v}_n(t) - \mathrm{v}(t), h)_H| = 0$$

for all $h \in H$. Notice that $v(t) \in H$ for any t if $v \in C([0,T]; H_{\mathrm{w}})$.
Our aim is to find a martingale solution to (6.12). By this we mean a weak solution in the probabilistic sense, according to the following

Definition 6.2. [solution to the martingale problem] We say that there exists a martingale solution of (6.12) if there exist

- a stochastic basis $(\hat{\Omega}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$
- a Y-cylindrical Wiener process $\hat{w}$

- a progressively measurable process $v : [0,T] \times \hat{\Omega} \to H$ with $\hat{\mathbb{P}}$-a.e. path
$$v \in C([0,T]; H_{\rm w}) \cap L^2(0,T;L^4)$$
and for any $t \in [0,T], \psi \in H^2$

$$\langle v(t), \psi\rangle + \int_0^t \langle Av(s), \psi\rangle ds + \int_0^t \langle B(v(s), v(s)), \psi\rangle ds \\ = \langle {\rm v}_0, \psi\rangle + \int_0^t \langle f(s), \psi\rangle ds + \langle \int_0^t G(v(s))\, d\hat{w}(s), \psi\rangle \tag{6.13}$$

$\hat{\mathbb{P}}$-a.s.

The regularity of the paths of this solution makes all the terms in (6.13) well defined, thanks to (6.10) and (6.14), i.e.

$$\mathbb{E}\left\| \int_0^t G(v(s))\, dw(s)\right\|^m_{H^{-g}} \le C_m (K_{g,2})^m t^{m/2}. \tag{6.14}$$

6.3. **Proof of the existence of solutions.** Looking for martingale solutions for system (6.12) one cannot use Itô calculus in the space H, since the covariance of the noise is not regular enough. Therefore, we introduce an approximating system by regularizing the covariance of the noise; this gives a sequence of approximating processes $\{{\rm v}_n\}_n$. In order to pass to the limit as $n \to \infty$ we need the tightness of the sequence of their laws. This is obtained by working with two auxiliary processes z_n and u_n with ${\rm v}_n = z_n + u_n$ in a similar way to [25].
Therefore, we first introduce a smoother problem which approximates (6.12); then we prove the tightness of the sequence of the laws; finally we show the convergence, providing existence of a martingale solution to (6.12).

6.3.1. *The approximating equation.* We start by defining the sequences

$$R_n = n(nI + A)^{-1} \qquad G_n = R_n G, \qquad n = 1, 2, \dots$$

We have, see [38], Sect. 1.3, that each R_n is a contraction operator in $H^{s,p}_{\rm sol}$ and it converges strongly to the identity operator, i.e.

$$\|R_n\|_{\mathcal{L}(H^{s,p}_{\rm sol}; H^{s,p}_{\rm sol})} \le 1 \qquad \text{and} \qquad \lim_{n\to\infty} R_n h = h \qquad \forall h \in H^{s,p}_{\rm sol}.$$

Moreover, it is a linear bounded operator from $H^{s,p}_{\rm sol}$ to $H^{s+t,p}_{\rm sol}$ for any $t \le 2$; but the operator norm is not uniformly bounded in n for $t > 0$. We point out the case for $p = 2$, needed in the sequel; we have

$$\|R_n\|_{\mathcal{L}(H^s_{\rm sol}; H^{s+t}_{\rm sol})} = \sup_{\|u\|_{H^s_{\rm sol}} \le 1} \|R_n u\|_{H^{s+t}_{\rm sol}}$$

and denoting by $\hat{u} = \hat{u}(\xi)$ the Fourier transform of u

$$\|R_n u\|_{H^{s+t}_{\rm sol}} = \|J^t R_n J^s u\|_{L^2} = \|n \frac{(1+|\xi|^2)^{\frac{t}{2}}}{n + |\xi|^2} (1+|\xi|^2)^{\frac{s}{2}} \hat{u}(\xi)\|_{L^2} \le B_n \|u\|_{H^s_{\rm sol}} \tag{6.15}$$

with $B_n := \|n\frac{(1+|\xi|^2)^{\frac{t}{2}}}{n+|\xi|^2}\|_{L_\infty} = C_t \frac{n}{(n-1)^{1-\frac{t}{2}}}$, given $t \in (0,2)$. For large n, the quantity B_n behaves like $n^{\frac{t}{2}}$, which is not bounded.
From the above and Baxendale [2]

$$\|G_n(\mathrm{v})\|_{\gamma(Y;H^{-g,p}_{\mathrm{sol}})} \le \|G(\mathrm{v})\|_{\gamma(Y;H^{-g,p}_{\mathrm{sol}})} \qquad \forall n \tag{6.16}$$

and

$$\lim_{n\to\infty} \|G_n(\mathrm{v}) - G(\mathrm{v})\|_{\gamma(Y;H^{-g,p}_{\mathrm{sol}})} = 0. \tag{6.17}$$

On the other hand, the operator $G_n(\mathrm{v})$ is more regular than $G(\mathrm{v})$. Indeed, assuming **(G1)**, $G_n(\mathrm{v})$ is a Hilbert-Schmidt operator in H, i.e.

$$\begin{aligned} \|G_n(\mathrm{v})\|_{\gamma(Y;H)} &\le \|R_n J^g\|_{\mathcal{L}(H;H)} \|J^{-g} G(\mathrm{v})\|_{\gamma(Y;H)} \\ &\le \|R_n\|_{\mathcal{L}(H;H^g)} \|G(\mathrm{v})\|_{\gamma(Y;H^{-g})}. \end{aligned} \tag{6.18}$$

For any $n \in \mathbb{N}$, we consider the following auxiliary problem

$$\begin{cases} d\mathrm{v}(t) + [A\mathrm{v}(t) + B(\mathrm{v}(t),\mathrm{v}(t))]dt = G_n(\mathrm{v}(t))\, dw(t) + f(t)dt, \ t \in (0,T] \\ \mathrm{v}(0) = \mathrm{v}_0 \end{cases} \tag{6.19}$$

This is the Navier-Stokes equation (6.12) with a more regular noise. Thanks to (6.18), the operator G_n is regular enough and one can prove that there exists a martingale solution for each n. The result is obtained by means of Itô formula for $d\|\mathrm{v}_n(t)\|^2_H$, as in Theorem 5.1 of [11] or Theorem 2.1 of [34]. More precisely

Proposition 6.3. *Let* $\mathrm{v}_0 \in H$ *and* $f \in L^p(0,T;H^{-1})$ *for some* $p > 2$. *If the assumption* ***(G1)*** *is satisfied, then for each n there exists a martingale solution* $((\Omega_n, \mathbb{F}_n, \mathbb{P}_n), w_n, \mathrm{v}_n)$ *of* (6.19)*; in addition, there exists a constant* C_n*, depending also on T,* $\|f\|_{L^p(0,T;H^{-1})}$, $\sup_{v\in H} \|G_n(v)\|_{\gamma(Y;H)}$*, such that*

$$\mathbb{E}_n \left[\sup_{0\le t\le T} \|\mathrm{v}_n(t)\|^2_H + \int_0^T \|\nabla \mathrm{v}_n(t)\|^2_{L^2} dt \right] \le C_n. \tag{6.20}$$

Moreover, if $d = 2$ *then* $\mathrm{v}_n \in C([0,T];H)$ $\mathbb{P}_n$*-a.s.*

6.3.2. *Tightness.* The estimates (6.20) are not uniform with respect to n, since (6.15) and (6.18) show that the Hilbert-Schmidt norms of the $G_n(v)$ are not uniformly bounded. Therefore from (6.20) one cannot get the tightness of the sequence of the laws of v_n. For this reason, in order to find suitable uniform estimates for the sequence $(u-n)_n$ we will follow a different path, which in fact has been used in [25] as well as other works, see for instance [8]. To be precise, we split our problem in two subproblems in the unknowns z_n and u_n with $\mathrm{v}_n = u_n + z_n$. Given the processes v_n and W_n from Proposition 6.3, we define the process z_n as the solution of the Ornstein-Uhlenbeck equation

$$dz_n(t) + Az_n(t)\, dt = G_n(\mathrm{v}_n(t))\, dw_n(t), \ t \in (0,T]; \qquad z_n(0) = 0. \tag{6.21}$$

Therefore the process $u_n = \mathrm{v}_n - z_n$ solves

$$\frac{du_n}{dt}(t) + Au_n(t) + B(\mathrm{v}_n(t),\mathrm{v}_n(t)) = f(t), \ t \in (0,T]; \qquad u_n(0) = \mathrm{v}_0 \tag{6.22}$$

We will first analyze the Ornstein-Uhlenbeck processes $(z_n)_n$ which satisfies the following identity

$$z_n(t) = \int_0^t e^{-(t-s)A} G_n(\mathrm{v}_n(s))\, dw_n(s). \tag{6.23}$$

We have two regularity results. The first of them shows the difference between our approach and that of [25], as we employ properties of the Itô stochastic integral in the 2-smooth Banach spaces as L^4 and $H^{\varepsilon,4}_{\mathrm{sol}}$). The second result also depends on this different approach as it is based on certain result from [6] established 2-smooth Banach spaces.

Lemma 6.4. *Assume conditions* **(G1)** *and* **(G2)***. Take any* $g_0 \in [g, 1)$ *and put* $\varepsilon = g_0 - g \geq 0$*. Then for any integer* $m \geq 2$ *there exists a constant* C *independent of* n *(but depending on* m*,* T *and* g_0*) such that*

$$\mathbb{E}_n \|z_n\|^m_{L^m(0,T;H^{\varepsilon,4}_{\mathrm{sol}})} \leq (K_{g,4} M)^m C.$$

In particular $z_n \in L^m(0, T; H^{0,4}_{\mathrm{sol}})$ $\mathbb{P}_n$*-a.s.*

Proof. We use Proposition 6.3 and Burkholder inequality (4.3) together with the fact that **(G2)** and (6.16) imply that $J^{-g} G_n(\mathrm{v}_n(s)) \in \gamma(Y; H^{0,4}_{\mathrm{sol}})$, for any $s \in [0, T]$. Finally we use results from [6] described earlier.

□

For $0 < \beta < 1$ let $C^\beta([0, T]; H^\delta)$ be the Banach space of H^δ-valued β-Hölder continuous functions endowed with the following norm

$$\|z\|_{C^\beta([0,T];H^\delta)} = \sup_{0\leq t\leq T} \|z(t)\|_{H^\delta} + \sup_{0\leq t<s\leq T} \frac{\|z(t) - z(s)\|_{H^\delta}}{|t - s|^\beta}.$$

Lemma 6.5. *Assume* **(G1)** *and let*

$$0 \leq \beta < \frac{1 - g}{2}. \tag{6.24}$$

Then, for any $p \geq 2$ *and* $\delta \geq 0$ *such that*

$$\beta + \frac{\delta}{2} + \frac{1}{p} < \frac{1 - g}{2} \tag{6.25}$$

there exists a modification $\tilde{z}_n$ *of* z_n *such that*

$$\mathbb{E}_n \|\tilde{z}_n\|^p_{C^\beta([0,T];H^\delta)} \leq \tilde{C} \tag{6.26}$$

for some constant $\tilde{C}$ *independent of* n *(but depending on* T*,* β*,* δ *and* p*).*

Proof. This follows from Corollary 3.5 of [6] described earlier. Details are as follows. Let us fix β, $p \geq 2$ and $\delta \geq 0$ as in conditions (6.24) and (6.25). Then there exists a modification $\tilde{z}_n$ of z_n such that

$$\mathbb{E}_n \|\tilde{z}_n\|^p_{C^\beta([0,T];H^\delta)} \leq C \mathbb{E}_n \int_0^T \|G_n(v_n(s))\|^p_{\gamma(Y;H^{-g})} ds. \tag{6.27}$$

Using (6.16) and assumption **(G1)** we conclude the proof of (6.26). □

In what follow we will collect some easy consequences of the previous two fundamental results.
Firstly, taking $\delta = 0, \beta < \frac{1-g}{2}$ and p big enough in (6.27), we get

$$\mathbb{E}_n \|z_n\|_{C^\beta([0,T];H)} \leq \left(\mathbb{E}_n \|z_n\|^p_{C^\beta([0,T];H)} \right)^{\frac{1}{p}} \leq (CT)^{\frac{1}{p}} K_{g,2}. \tag{6.28}$$

Secondly, taking $\delta = \frac{1-g}{2}, \beta = 0$ and p big enough in (6.27), we get

$$\mathbb{E}_n \|z_n\|_{C([0,T];H^{\frac{1-g}{2}})} \leq \left(\mathbb{E}_n \|z_n\|^p_{C([0,T];H^{\frac{1-g}{2}})} \right)^{\frac{1}{p}} \leq (CT)^{\frac{1}{p}} K_{g,2}. \tag{6.29}$$

Hence, a consequence of the above two lemmas is that there exist finite constants E_m, E_g and $E_{\beta,\delta}$ such that

$$\sup_n \mathbb{E}_n \|z_n\|^m_{L^m(0,T;H^{0,4}_{\mathrm{sol}})} = (E_m)^m \tag{6.30}$$

$$\sup_n \mathbb{E}_n \|z_n\|_{C([0,T];H^{\frac{1-g}{2}})} = E_g \tag{6.31}$$

$$\sup_n \mathbb{E}_n \|z_n\|^p_{C^\beta([0,T];H^\delta)} = (E_{\beta,\delta})^p \tag{6.32}$$

where β and δ are as in Lemma 6.5.
We recall that each u_n solves a deterministic equation with two forcing terms: one is random and the other is deterministic:

$$\frac{du_n}{dt}(t) + Au_n(t) = -B(\mathrm{v}_n(t), \mathrm{v}_n(t)) + f(t) \tag{6.33}$$

with $u_n(0) = \mathrm{v}_0$. We analyze the above equation (6.33) pathwise.
We have the following fundamental results about the uniform estimates of the approximating processes (u_n).

Proposition 6.6. *Assume* ***(G1)*** *and* ***(G2)****. Let* $\mathrm{v}_0 \in H$ *and* $f \in L^p(0,T;H^{-1})$ *for some* $p > 2$*. Put*

$$Z_T = L^\infty(0,T;H) \cap L^2(0,T;H^1) \cap L^{\frac{4}{1-g}}(0,T;H^{\frac{1-g}{2}}) \cap L^{\frac{8}{d}}(0,T;L^4) \cap C^{1-\frac{d}{4}}([0,T];H^{-1})$$

Then, for any n the paths of the process $u_n = \mathrm{v}_n - z_n$ *solving* (6.33) *are such that*

$$u_n \in Z_T, \ \mathbb{P}_n\text{-a.s.},$$

and for every $\varepsilon > 0$ *there exists a positive constant* $R > 0$ *such that*

$$\sup_n \mathbb{P}_n(\|u_n\|_{Z_T} > R) \leq \varepsilon.$$

Now we need to apply a tightness argument in order to pass to the limit. Merging the estimates (6.30)-(6.32) for the process z_n and those for u_n from Proposition 6.6 we get estimates for $\mathrm{v}_n = z_n + u_n$. These estimates in probability are uniform with respect to n.

Proposition 6.7. *Assume* ***(G1)*** *and* ***(G2)****, let* $\mathrm{v}_0 \in H$ *and* $f \in L^p(0,T;H^{-1})$ *for some* $p > 2$ *and let* $((\Omega_n, \mathbb{F}_n, \mathbb{P}_n), w_n, \mathrm{v}_n)$ *be a martingale solution of* (6.19) *as given in Proposition 6.3.*

Then there exist $\gamma, \delta > 0$ such that for any $\varepsilon > 0$ there exists a positive constant $R > 0$ such that

$$\sup_n \mathbb{P}_n(\|v_n\|_{\hat{Z}_T} > R) \leq \varepsilon$$

where

$$\hat{Z}_T := L^\infty(0,T;H) \cap L^2(0,T;H^\delta) \cap L^{\frac{4}{1-g}}(0,T;H^{\frac{1-g}{2}}) \cap L^{\frac{8}{d}}(0,T;L^4) \cap C^\gamma([0,T];H^{-1}).$$

Actually, $\gamma = \min(\beta, 1 - \frac{d}{4})$ with β and δ fulfilling (6.25). Therefore $0 < \gamma < \frac{1}{2}$ and $0 < \delta < 1$.

6.3.3. *Convergence.* We will use the following Jakubowski's version of the Skorokhod Theorem in the form given by Brzeźniak and Ondreját [13], see also [27].

Theorem 6.8. (Theorem A.1 in [13]) *Let $\mathcal{X}$ be a topological space such that there exists a sequence $\{f_m\}$ of continuous functions $f_m : \mathcal{X} \to \mathbb{R}$ that separates points of $\mathcal{X}$. Let us denote by $\mathcal{S}$ the σ-algebra generated by the maps $\{f_m\}$. Then*

- **(j1) :** *every compact subset of $\mathcal{X}$ is metrizable,*
- **(j2) :** *every Borel subset of a σ-compact set in $\mathcal{X}$ belongs to $\mathcal{S}$,*
- **(j3) :** *every probability measure supported by a σ-compact set in $\mathcal{X}$ has a unique Radon extension to the Borel σ-algebra on $\mathcal{X}$,*
- **(j4) :** *if (μ_m) is a tight sequence of probability measures on $(\mathcal{X}, \mathcal{S})$, then there exists a subsequence (m_k), a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{X}$-valued Borel measurable variables ξ_k, ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω. Moreover, the law of ξ is a Radon measure.*

We will also need

Lemma 6.9 (tightness criterion)**.** *We are given parameters $\gamma > 0$, $\delta > 0$, $1 < p < \infty$ and a sequence $\{v_n\}_{n\in\mathbb{N}}$ of adapted processes in $C([0,T];H^{-1})$.*
Assume that for any $\varepsilon > 0$ there exist positive constant $R > 0$ such that

$$\sup_n \mathbb{P}(\|v_n\|_{L^p(0,T;L^4)} > R_1) \leq \varepsilon \tag{6.34}$$

$$\sup_n \mathbb{P}(\|v_n\|_{C^\gamma([0,T];H^{-1})} > R_2) \leq \varepsilon \tag{6.35}$$

$$\sup_n \mathbb{P}(\|v_n\|_{L^2(0,T;H^\delta)} > R_3) \leq \varepsilon \tag{6.36}$$

$$\sup_n \mathbb{P}(\|v_n\|_{L^\infty(0,T;H)} > R_4) \leq \varepsilon \tag{6.37}$$

Let

$$Z_T = L^p_{\rm w}(0,T;L^4) \cap C([0,T];U') \cap L^2(0,T;H_{\rm loc}) \cap C([0,T];H_{\rm w})$$

and let μ_n be the law of v_n on Z_T. Then, the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is tight in Z.

The latter result provides the tightness to pass to the limit. We have

Theorem 6.10. *Let $v_0 \in H$ and $f \in L^p(0,T;H^{-1})$ for some $p > 2$. If **(G1)**- **(G2)** are satisfied, then there exists a martingale solution $\big((\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}), \tilde{w}, \tilde{v}\big)$ of (6.12); in addition*

$$\tilde{v} \in L^{\frac{4}{1-g}}(0,T;H^{\frac{1-g}{2}}) \cap L^{\frac{8}{d}}(0,T;L^4) \qquad \tilde{\mathbb{P}} - a.s. \tag{6.38}$$

Moreover, if $d = 2$ then $\tilde{v} \in C([0,T]; H)$ $\tilde{\mathbb{P}}$-a.s.

Proof. One proceeds as in [11].
We fix $0 < \gamma < \frac{1}{2}$ and $0 < \delta < 1$ appearing in Proposition 6.7 and define the space

$$Z = L_{\mathrm{w}}^{\frac{8}{d}}(0,T; L^4) \cap C([0,T]; U') \cap L^2(0,T; H_{\mathrm{loc}}) \cap C([0,T]; H_{\mathrm{w}})$$

with the topology $\mathcal{T}$ given by the supremum of the corresponding topologies. According to Lemma 6.9, Proposition 6.7 provides that the sequence of laws of the processes v_n is tight in Z.
By the Jakubowski's generalization of the Skohorokod Theorem to nonmetric spaces, see [11] and [27], there exist a subsequence $\{\mathrm{v}_{n_k}\}_{k=1}^{\infty}$, a stochastic basis $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, Z-valued Borel measurable variables $\tilde{\mathrm{v}}$ and $\{\tilde{\mathrm{v}}_k\}_{k=1}^{\infty}$ such that for any k the laws of v_{n_k} and $\tilde{\mathrm{v}}_k$ are the same and $\tilde{\mathrm{v}}_k$ converges to $\tilde{\mathrm{v}}$ $\tilde{\mathbb{P}}$-a.s. with the topology $\mathcal{T}$.
Since each $\tilde{\mathrm{v}}_k$ has the same law as v_{n_k}, it is a martingale solution to equation (6.19); therefore each process

$$\tilde{M}_k(t) := \tilde{\mathrm{v}}_k(t) - \tilde{\mathrm{v}}_k(0) + \int_0^t A\tilde{\mathrm{v}}_k(s)ds + \int_0^t B(\tilde{\mathrm{v}}_k(s), \tilde{\mathrm{v}}_k(s))ds - \int_0^t f(s)ds$$

is a martingale with quadratic variation

$$\langle\langle\, \tilde{M}_k \,\rangle\rangle(t) = \int_0^t G_k(\tilde{\mathrm{v}}_k(s))G_k(\tilde{\mathrm{v}}_k(s))^* ds.$$

It is now classical to show, see e.g. [11], that

$$\langle \tilde{M}_k(t) - \tilde{M}(t), \phi \rangle \to 0$$

for any $\phi \in H^2$ and every $t \in [0,T]$, where

$$\tilde{M}(t) = \tilde{\mathrm{v}}(t) - \tilde{\mathrm{v}}(0) + \int_0^t A\tilde{\mathrm{v}}(s)ds + \int_0^t B(\tilde{\mathrm{v}}(s), \tilde{\mathrm{v}}(s))ds - \int_0^t f(s)ds.$$

There is no convergence the quadratic variation processes, but the quadratic variation of the more regular process $J^{-g}\tilde{M}_k$

$$\langle\langle\, J^{-g}\tilde{M}_k \,\rangle\rangle(t) = \int_0^t J^{-g}G_k(\tilde{\mathrm{v}}_k(s))G_k(\tilde{\mathrm{v}}_k(s))^* J^{-g} ds.\text{‘}$$

is finite and thanks to (6.17) it converges to

$$\int_0^t J^{-g}G(\tilde{\mathrm{v}}(s))G(\tilde{\mathrm{v}}(s))^* J^{-g} ds$$

as $n \to \infty$.
With the usual **Martingale Representation Theorem**, see e.g. [18], there exist

- a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, where $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t \geq 0}$,
- a Y-cylindrical Wiener process $\tilde{W}(t)$ defined on this basis,
- and a progressively measurable process $\tilde{\mathrm{v}}(t)$ such that

$$J^{-g}\tilde{M}(t) = \int_0^t J^{-g}G(\tilde{\mathrm{v}}(s))\, d\tilde{w}(s).$$

Therefore $\tilde{\mathrm{v}}$ is a martingale solution to (6.12).
Finally, (6.38) comes from the uniform estimates of Proposition 6.7.

□

7. Lecture 8: Variational theory of linear SPDEs in Hilbert space

The aim of this section is to present the celebrated result of Pardoux [37] and Krylov-Rosovski [30] on stochastic parabolic equations driven by Wiener process. To put our results into right framework let us recall the result of Pardoux and Krylov-Rosovski. We use the formulation of the former author. Suppose that

$$V \subset H = H' \subset V'$$

is a Gelfand triple of Hilbert spaces. We will study the following equation

$$\begin{aligned} &du(t) + A(t)u(t)\,dt - C(t)u(t)dW(t) = f(t)\,dt + g(t)\,dW(t), \quad t \geq 0, \\ &u(0) = u_0. \end{aligned} \tag{7.1}$$

We suppose that K is a real separable Hilbert space, $A(t)$, $C(t)$, $t \in [0,T]$ are two families of linear operators satisfying the following assumptions

$$A \in L^\infty(0,T;\mathcal{L}(V,V')), \tag{7.2}$$

$$B \in L^\infty(0,T;R(K,H)), \tag{7.3}$$

where $R(H,H)$ is the space of all γ-radonifying, i.e. Hilbert-Schmidt, operators form K to H, and the following coercivity assumption.

Assumption 7.1. *There exists $\nu > 0$ and $\lambda \in \mathbb{R}$ such that for a.a. $t \in [0,T]$,*

$$\langle A(t)u,u\rangle + \lambda|u|^2 \geq \nu\|u\|^2 + \frac{1}{2}\,\|C(t)u\|^2_{R(K,H)}, \quad u \in V. \tag{7.4}$$

In the above, $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_{R(K,H)}$ denote respectively the norm in V, H and $R(K,H)$. By $\langle\cdot,\cdot\rangle$ we denote the duality between V' and V, while the inner products in V and H will be denoted respectively by $(\cdot,\cdot)_V$ and respectively $(\cdot,\cdot)_H$.
Moreover, $W(t)$, $t \geq 0$, is a canonical K-cylindrical Wiener process defined on some fixed complete filtered probability space. Moreover, $f(t)$, $t \in [0,T]$ and $g(t)$, $t \in [0,T]$ are progressively measurable V' and resp. $R(K,H)$-valued processes such that

$$\mathbb{E}\int_0^T |f(t)|^2_{V'}\,dt < \infty, \tag{7.5}$$

$$\mathbb{E}\int_0^T \|g(t)\|^2_{R(K,H)}\,dt < \infty. \tag{7.6}$$

Suppose finally, that u_0 belongs to $L^2(\Omega,\mathcal{F}_0;H)$. Under the above assumptions Pardoux, see [37, Theorem 1.3], proved

Theorem 7.2. *There exists a unique progressively measurable process $u(t)$ such that u is a solution to problem* (7.1) *and moreover*

$$\mathbb{E}\int_0^T \|u(t)\|^2\,dt < \infty \tag{7.7}$$

$$u \in L^2(\Omega; C(0,T;H)) \tag{7.8}$$

$$|u(t)|^2 + 2\int_0^t \langle A(s)u(s), u(s)\rangle\,ds = |u_0|^2 + 2\int_0^t (g(s) + C(s)u(s), u(s))dW(s) \tag{7.9}$$
$$+ \int_0^t (u(s), f(s)\,ds + \int_0^t \|C(s)u(s) + g(s)\|^2_{R(K,H)}\,ds, \quad a.s..$$

Our aim is this paper is to generalise this result in the following sense.
Suppose that τ is a accessible stopping time and let τ_u be a certain increasing sequence of stopping times $\mathbb{P}$-a.s. convergent to τ. Assume that $f(t)$, $t \in [0,\tau)$ and $g(t)$, $t \in [0,\tau)$ are progressively measurable V' and resp. $R(K,H)$-valued processes such that for each $n \in \mathbb{N}$,

$$\mathbb{E}\int_0^{\tau_n} |f(t)|^2_{V'}\,dt < \infty, \tag{7.10}$$

$$\mathbb{E}\int_0^{\tau_n} \|g(t)\|^2_{R(K,H)}\,dt < \infty. \tag{7.11}$$

First we shall generalise [37, Theorem 1.2].

Theorem 7.3. *In addition to the above assumption let us assume that $u(t)$, $t \in [0,\tau)$ is a progressively measurable V-valued process such that for each $n \in \mathbb{N}$,*

$$\mathbb{E}\int_0^{\tau_n} |u(t)|^2_V\,dt < \infty, \tag{7.12}$$

$$\mathbb{E}\sup_{t\in[0,\tau_n]} |u(t)|^2_H\,dt < \infty. \tag{7.13}$$

Suppose also that $\psi : H \to \mathbb{R}$ is a twice Fréchet differentiable function such that

(i) *ψ, ψ' and ψ'' are bounded on balls,*
(iii) *for each operator $Q \in \mathcal{T}_1(H)$, the function $H \ni x \mapsto \mathrm{tr}_H(Q \circ \psi''(x)) \in \mathbb{R}$ is continuous,*
(iv) *for each $x \in V$, the restriction of $d_x\psi = \psi'(x)$ to the space V is continuous and, if $\nabla_V\psi(x)$ denotes the unique element in V such that*

$$(d_x\psi)(y) = (\nabla_V\psi(x), y)_V, \ y \in V,$$

then the map $V \ni x \mapsto \nabla_V\psi(x) \in V$ is (V,s)-(V,w) continuous, where (V,s), resp. (V,w) denotes the space V endowed with the strong, resp. weak, topology;
(v) *the map $V \ni x \mapsto \nabla_V\psi \in V$ is of linear growth, i.e. there exists $k > 0$ such that $\|\nabla_V\psi(x)\| \le k(1 + \|x\|)$, $x \in V$.*

Suppose that that u is a local solution of the problem (7.1)*, i.e. for each $n \in \mathbb{N}$, for all $t \ge 0$, $\mathbb{P}$-a.s.,*

$$u(t\wedge\tau_n)) = u(0)+\int_0^{t\wedge\tau_n} [C(s)u(s)+g(s)]\,dW(s)+\int_0^{t\wedge\tau_n} [-Au(s)+f(s)]\,ds. \tag{7.14}$$

Then, for each $n \in \mathbb{N}$, for all $t \geq 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\psi(u(t\wedge\tau_n)) &= \psi(u(0)) + 2\int_0^{t\wedge\tau_n} (\nabla_H\psi(u(s)), [C(s)u(s)+g(s)]dW(s))_H \\ &+ \int_0^{t\wedge\tau_n} \langle -A(s)u(s)+f(s), \nabla_V(u(s))\rangle\,ds + \int_0^{t\wedge\tau_n} \mathrm{tr}_{g(s)}\psi''(u(s))\,ds. \end{aligned}\tag{7.15}$$

Remark 7.4. $\nabla_H\psi(x)$ denotes the unique element in H such that

$$(d_x\psi)(y) = (\nabla_H\psi(x), y)_H,\ \ y \in H$$

$\psi'' : H \to \mathcal{L}(H, \mathcal{L}(H,\mathbb{R})) \cong \mathcal{L}(H, H; \mathbb{R}) = \mathbb{L}(H, H; \mathbb{R})$.

Proof. The proof of the above result is a combination of the proof of [37, Theorems 1.2 and 1.3] and the approximation argument from ☐

In particular, with function $\psi(x) = |x|^2$, $x \in H$, we have the following result.

Corollary 7.5. *Suppose that τ is a accessible stopping time and let τ_u be a certain increasing sequence of stopping times $\mathbb{P}$-a.s. convergent to τ. Assume that $f(t)$, $t \in [0,\tau)$ and $g(t)$, $t \in [0,\tau)$ are progressively measurable V' and resp. $R(K,H)$-valued processes such that for each $n \in \mathbb{N}$, they satisfy the conditions* (7.10) *and* (7.11)*. Suppose that $u(t)$, $t \in [0,\tau)$ is a progressively measurable V-valued process such that for each $n \in \mathbb{N}$, it satisfies the conditions* (7.12) *and* (7.13)*. Suppose finally that u is a local solution of the problem* (7.1)*. Then, for each $n \in \mathbb{N}$, for all $t \geq 0$, $\mathbb{P}$-a.s.,*

$$\begin{aligned}|u(t\wedge\tau_n)|^2 &+ 2\int_0^{t\wedge\tau_n} \langle A(s)u(s), u(s)\rangle\,ds \\ &= |u(0)|^2 + 2\int_0^{t\wedge\tau_n} (u(s), (g(s)+C(s)u(s))dW(s)) \\ &+ \int_0^{t\wedge\tau_n} \langle f(s), u(s)\rangle\,ds + \int_0^{t\wedge\tau_n} \|C(s)u(s)+g(s)\|^2_{R(K,H)}\,ds.\end{aligned}\tag{7.16}$$

Assume now that we have also a nonnegative progressively measurable process $a(t)$, $0 \leq t < \tau$ and we define another nonnegative process y by

$$y(t) := e^{-a(t)},\ \ 0 \leq t < \tau.$$

Then $dy(t) = -a(t)y(t)\,dt$ and by the chain rule we infer that the process $y(t)|u(t)|^2$, $0 \leq t < \tau$ satisfies the following

$$\begin{aligned}d[y(t)|u(t)|^2] &= y(t)\,d[|u(t)|^2] + |u(t)|^2\,dy(t) \\ &= y(t)(u(t), (g(t)+C(t)u(t))\,dW(t)) + y(t)[-2\langle A(t)u(t), u(t)\rangle \\ &+ \langle f(t), u(t)\rangle + \|C(t)u(t)+g(t)\|^2_{R(K,H)} - a(t)|u(t)|^2]\,dt\end{aligned}$$

Let us now assume that the processes f and g are of special form. To be precise, let us assume that
There exist $\alpha,\beta > 0$ such that $\alpha+\beta<1$, there exist $C_1, C_2 > 0$ and there exists a nonnegative progressively measurable process $\varphi(t)$, $0\leq t<\tau$, such that

$$|\langle f(t), u(t)\rangle \;\leq\; \alpha\nu\|u(t)\|^2 + \frac{C_1}{\nu}\varphi(t)|u(t)|^2, \tag{7.17}$$

$$\|C(t)u(t)+g(t)\|^2_{R(K,H)} \;\leq\; \beta\nu\|u(t)\|^2 + \frac{C_2}{\nu}\varphi(t)|u(t)|^2, \tag{7.18}$$

Then, we have

$$\begin{aligned}
d[y(t)|u(t)|^2] \;&+\; 2y(t)\langle A(t)u(t),u(t)\rangle - y(t)(u(t),(g(t)+C(t)u(t))\,dW(t))\\
&=\; y(t)[\langle f(t),u(t)\rangle + \|C(t)u(t)+g(t)\|^2_{R(K,H)} - a(t)|u(t)|^2]\,dt\\
&\leq\; (\alpha+\beta)\nu y(t)\|u(t)\|^2 + [\frac{C_1+C_2}{\nu}\varphi(t) - a(t)]y(t)|u(t)|^2,
\end{aligned}$$

Applying next assumption (7.4) we infer that

$$\begin{aligned}
d[y(t)|u(t)|^2] \;&+\; \nu y(t)\|u(t)\|^2 + \frac{1}{2}\,y(t)\|C(t)u(t)\|^2_{R(K,H)}\\
&-\; y(t)(u(t),(g(t)+C(t)u(t))\,dW(t))\\
&\leq\; (\alpha+\beta)\nu y(t)\|u(t)\|^2 + [\frac{C_1+C_2}{\nu}\varphi(t)+\lambda - a(t)]y(t)|u(t)|^2,
\end{aligned}$$

Therefore, with $\delta=\nu(1-\alpha-\beta)$ and $C_3=\frac{C_1+C_2}{\nu}$ we infer that for all $n\in\mathbb{N}$,

$$\begin{aligned}
d[y(t)|u(t)|^2] \;&+\; \delta y(t)\|u(t)\|^2 - y(t)(u(t),(g(t)+C(t)u(t))\,dW(t))\\
&\leq\; [C_3\varphi(t)+\lambda-a(t)]y(t)|u(t)|^2,
\end{aligned}$$

Hence, for all $n\in\mathbb{N}$,

$$\begin{aligned}
y(t\wedge\tau_n)|u(t\wedge\tau_n)|^2 \;&+\; \delta\int_0^{t\wedge\tau_n} y(s)\|u(s)\|^2\,ds\\
&-\; \int_0^{t\wedge\tau_n} y(s)(u(s),(g(s)+C(s)u(s))\,dW(s))\\
&\leq\; y(0)|u(0)|^2 + \int_0^{t\wedge\tau_n}[C_3+\lambda-a(t)]y(t)\varphi(s)|u(s)|^2\,ds
\end{aligned}$$

Since the process $\int_0^{t\wedge\tau_n} y(s)(u(s),(g(s)+C(s)u(s))\,dW(s))$ is a martingale, we get, by taking the expectation, that

$$\begin{aligned}
\mathbb{E}[y(t\wedge\tau_n)|u(t\wedge\tau_n)|^2] \;&+\; \delta\mathbb{E}\int_0^{t\wedge\tau_n} y(s)\|u(s)\|^2\,ds \leq \mathbb{E}[y(0)|u(0)|^2]\\
&+\; \mathbb{E}\int_0^{t\wedge\tau_n}[C_3+\lambda-a(t)]y(t)\varphi(s)|u(s)|^2\,ds
\end{aligned}$$

8. Lecture 9: Applications to stochastic Schrödinger Equations (via B+Millet), existence of strong solutions

First we study the existence and uniqueness of solutions to appropriate approximated problems. Then we prove the existence of a maximal solution. Note that even if the lifetime τ_∞ of the solution is defined in terms of the sum of two norms, we prove that the $H^{1,2}$ norm of $u(t)$ explodes as $t \nearrow \tau_\infty < \infty$. We also describe an abstract formulation of the NLS equation in Stratonovich form. As in [19] this is needed to obtain a global solution since the L^2-norm of the solution is preserved in this formulation.

We then restrict the framework as follows. We consider a 2-dimensional compact riemannian manifold M, a regular function $\tilde{g} : \mathbb{R} \to \mathbb{R}$ and the diffusion coefficient of the form $g(u) = \tilde{g}(|u|^2)u$. We establish the local existence and uniqueness of the solution to the specific NLS equation in Stratonovich form

$$idu(t) + \Delta u(t) = f(u(t))\, dt + \tilde{g}(|u(t)|^2)u(t) \circ dW(t).$$

Note that unlike the usual parabolic case, see [7], the Stratonovich correction term does not contain the derivative of g or of $\tilde{g}$.

Finally we will deal with the existence and uniqueness of a global solution for 2-dimensional manifold, an initial condition $u_0 \in H^{1,2}$, when the drift and the diffusion coefficients have the specific form $f(u) = \tilde{f}(|u|^2)u$ and $g(u) = \tilde{g}(|u|^2)u$ respectively (see Theorem 8.40). This is the natural framework to extend the deterministic well-posedeness proved in Burq et al in [16]. The function $\tilde{f}$ is either defocusing, that is a polynomial with a positive leading coefficient or $\tilde{f}(r) = Cr^\sigma$ with $C > 0$ and $\sigma \in [\frac{1}{2}, \infty)$, or $\tilde{f}$ is focusing, that is $\tilde{f}(r) = -Cr^\sigma$ with $\sigma \in [\frac{1}{2}, 1)$ and $C > 0$. The stochastic integral is defined in Stratonovich form and depending on the nonlinearity, some more conditions have to be imposed on the noise W. More precisely, W has to take values in a sub algebra $H^{1,2}(M) \cap H^{1,\alpha}(M)$ of the space $H^{1,2}(M) \cap W^{\hat{s},q}(M)$, with $\hat{s} = 1 - \frac{1}{p}$ and $\frac{2}{p} + \frac{2}{q} = 1$, for some p chosen from the non linear term f.

By $L^q(M)$, $q \in [1, \infty]$ we will denote the classical real Banach space of all [equivalence classes of] $\mathbb{C} \cong \mathbb{R}^2$-valued q-integrable functions on M, endowed with the classical norm which will be denoted by $|\cdot|_q$ (or sometimes, in danger of ambiguity by $|\cdot|_{L^q}$).

Let us recall a definition of the classical Sobolev spaces $H^{\theta,q}(M)$ and of the Besov-Slobodetski space $W^{\theta,q}(M)$. The former space is defined as the complex interpolation space
$[L^q(M), H^{k,q}(M)]_{\frac{\theta}{k}}$, where k is a natural number bigger that θ. It can be shown that $H^{\theta,q}(M) := D((-\Delta_q)^{\theta/2})$, where Δ_q is the Laplace-Beltrami operator on $L^q(M)$, i.e. the infinitesimal generator of the heat semigroup on the space $L^q(M)$. The latter space, defined by
(8.1)

$$W^{\theta,q}(M) := \left\{ f \in L^q(M) : |f|^q_{\theta,q} := \int_M \int_M \frac{|f(x_2) - f(x_1)|^q}{|x_2 - x_1|^{d+\theta q}}\, dx_1\, dx_2 < \infty \right\},$$

is endowed with the norm $\|f\|_{W^{\theta,q}(M)} := |f|_{L^q(M)} + \lfloor f \rfloor_{\theta,q}$. Note that $W^{\theta,2}(M) = H^{\theta,2}(M)$, whereas $H^{\theta,q}(M) \subset W^{\theta,q}(M)$ (or $H^{\theta,q}(M) \supset W^{\theta,q}(M)$) depending whether $q > 2$ (or $q < 2$). Let us recall that the Besov-Slobodetski spaces with fractional order $\theta \in (0,1)$ are equal to the real interpolation spaces of order θ between the spaces $L^q(M)$ and $H^{1,q}(M) = W^{1,q}(M)$; see for instance [40]. We will also use the notation

$$\lfloor f \rfloor_{1,q}^q := \int_M |\nabla f|^q \, dx.$$

Thus, $f \in W^{1,q}(M)$ iff $f \in L^q(M)$ and $\lfloor f \rfloor_{1,q} < \infty$.
Given a real separable Banach space Y, $\theta \in [0,1]$ and $q \in [1,\infty)$, let

$$\mathcal{R}^{\theta,q}(Y) = W^{\theta,q}(M,Y) \cap L^\infty(M,Y), \tag{8.2}$$

endowed with the norm $\|u\|_{\mathcal{R}^{\theta,q}(Y)} = \|u\|_{W^{\theta,q}(M,Y)} + |u|_{L^\infty(M,Y)}$. Once more to ease notations, let $\mathcal{R}^{\theta,q} = \mathcal{R}^{\theta,q}(M,\mathbb{C})$ where $\mathbb{C} \equiv \mathbb{R}^2$.

Proposition 8.1. *Fix $\theta \in (0,1]$ and $q \in [1,\infty)$. Let Y and Y_1 be real separable Banach spaces and $f : Y \to Y_1$ be Lipschitz on balls when $\theta < 1$, or, everywhere differentiable when $\theta = 1$. Then the Nemytski map F corresponding to f maps $\mathcal{R}^{\theta,q}(Y)$ to $\mathcal{R}^{\theta,q}(Y_1)$. More precisely, when $\theta < 1$, then for all $\gamma \in \mathcal{R}^{\theta,q}(Y)$,*

$$\|F(\gamma)\|_{\theta,q} \leq |f(0)|\mathrm{vol}(M) + K_1(f, |\gamma|_\infty)\|\gamma\|_{\theta,q}. \tag{8.3}$$

When $\theta = 1$, inequality (8.3) *holds with K_1 being replaced by $\tilde{K}_1$. In particular, F is of linear growth either if f is globally Lipschitz, i.e. $K_1(f) := \sup_{R>0} K_1(f,R)$ is finite when $\theta < 1$, or if f' is bounded, i.e. $\tilde{K}_1(f) := \sup_{R>0} \tilde{K}_1(f,R) < \infty$ when $\theta = 1$.*
The above statements remain true if the space $\mathcal{R}^{\theta,q}(Y)$ is replaced by $\tilde{\mathcal{R}}^{\theta,q}_{s,p}(Y) = H^{1,2}(M) \cap W^{s,p}(M)$, provided that $1 > s > \frac{d}{p}$.

Let us formulate the following important (but simple) consequence of Proposition 8.1.

Corollary 8.2. *Let $\theta \in (0,1]$ and $q \in [1,\infty)$. Then $\mathcal{R}^{\theta,q}$ is an algebra (with pointwise multiplication) and there exists a constant $C > 0$ such that for $\sigma, \gamma \in W^{\theta,q}(M) \cap L^\infty(M)$,*

$$\lfloor \sigma\gamma \rfloor_{\theta,q} \leq \|\sigma\gamma\|_{\theta,q} \leq |\sigma|_{L^\infty}\|\gamma\|_{\theta,q} + |\gamma|_{L^\infty}\|\sigma\|_{\theta,q} \leq C'\|\sigma\|_{W^{\theta,q}\cap L^\infty}|\gamma|_{W^{\theta,q}\cap L^\infty}. \tag{8.4}$$

Now we will formulate the promised generalisation of Proposition 8.1.

Theorem 8.3. *Fix $\theta \in (0,1]$ and $q \in [1,\infty)$ and let Y, Y_1 be real separable Banach spaces. Assume that a function $f : Y \to Y_1$ is of C^1 class and its Fréchet derivative $f' : Y \to \mathcal{L}(Y, Y_1)$ is Lipschitz on balls. Then the Nemytski map F corresponding to f is Lipschitz continuous on balls in $\mathcal{R}^{\theta,q}(Y)$. More precisely for any $K > 0$, and all $\gamma, \sigma \in \mathcal{R}^{\theta,q}(Y)$ with $\|\gamma\|_{\mathcal{R}^{\theta,q}(Y)} \vee \|\sigma\|_{\mathcal{R}^{\theta,q}(Y)} \leq K$, we have:*

$$|F(\gamma) - F(\sigma)|_q \leq K_1(f, |\gamma|_\infty \vee |\sigma|_\infty)\, |\gamma - \sigma|_q, \tag{8.5}$$

$$\begin{aligned} \lfloor F(\gamma) - F(\sigma) \rfloor_{\theta,q} \leq{}& K_2(f, |\gamma|_\infty \vee |\sigma|_\infty)\, |\gamma - \sigma|_\infty [\lfloor \sigma \rfloor_{\theta,q} + \frac{1}{2}\lfloor \gamma - \sigma \rfloor_{\theta,q}] \\ &+ K_1(f, |\gamma|_\infty \vee |\sigma|_\infty)\, \lfloor \gamma - \sigma \rfloor_{\theta,q}. \end{aligned} \tag{8.6}$$

The above statements remain true if the space $\mathcal{R}^{\theta,q}(Y)$ *is replaced by* $\tilde{\mathcal{R}}^{\theta,q}_{s,p}(Y) = H^{1,2}(M) \cap W^{s,p}(M)$*, provided that* $1 > s > \frac{d}{p}$.

Corollary 8.4. *Under the assumptions of Proposition 8.1 the map* $F : \mathcal{R}^{\theta,q}(Y) \to \mathcal{R}^{\theta,q}(Y_1)$ *is measurable.*

Remark 8.5. Let $Y = \mathbb{C} \equiv \mathbb{R}^2$ and let $f : Y \to Y$ be defined by $f(z) = C\,|z|^{2\alpha}\,z$ for some real constants $\alpha \geq \frac{1}{2}$ and C. Then f is of class C^1 and both f and f' are Lipschitz on balls. Furthermore, given $\sigma \geq \frac{3}{2}$, $\theta \in (0,1]$ and q such that $\theta d > q$, the map $\Phi :\in W^{\theta,q}(\mathbb{C}) \to \mathbb{R}$ defined by

$$\Phi(u) = \int_M |u(x)|^{2\sigma} dx$$

is of class C^2, and for $u, v_1, v_2 \in W^{\theta,q}(\mathbb{C})$, we have

$$\Phi'(u)(v_1) = \int_M 2\sigma |u(x)|^{2(\sigma-1)} Re(u(x)\overline{v_1(x)})dx$$

$$\Phi''(u)(v_1, v_2) = \int_M \Big[4\sigma(\sigma-1)|u(x)|^{2(\alpha-2)} Re(u(x)\overline{v_1(x)}) Re(u(x)\overline{v_2(x)}) + 2\sigma|u(x)|^{2(\sigma-1)} Re(v_2(x)\overline{v_1(x)})\Big]dx.$$

8.1. Stochastic Strichartz estimates.

We assume the following.

Assumption 8.6.(i) H_0 *is a separable Hilbert space and* E_0 *is a separable Banach space such that* $\mathrm{E}_0 \cap \mathrm{H}_0$ *is dense in both* E_0 *and* H_0*;*

(ii) *There exists a separable Hilbert space* $\mathcal{H}_0$ *such that* $\mathrm{H}_0 \subset \mathcal{H}_0$ *and a* C_0 *unitary group* $\mathbf{U} = (U_t)_{t\in\mathbb{R}}$ *on* $\mathcal{H}_0$ *with the infinitesimal generator iA, where A is self-adjoint in* $\mathcal{H}_0$.
Assume that the restriction of $-A$ *to* H_0*, denoted also by* $-A$*, is a positive operator in* H_0.

(iii) *There exists a positive linear operator* $-\tilde{A}$ *on the space* E_0 *such that* $D(A) \cap E_0 \subset D(\tilde{A})$, $D(\tilde{A}) \cap \mathrm{H}_0 \subset D(A)$ *and* $A = \tilde{A}$ *on* $D(A) \cap D(\tilde{A})$*. In what follows, unless in a danger of ambiguity, the operator* $\tilde{A}$ *will be denoted by A.*

(iv) *There exists a number* $p \in (2,\infty)$ *and a non decreasing function* $\tilde{C}_p : (0,\infty) \to (0,\infty)$ *such that* $\tilde{C}_p(0^+) = 0$ *and, for every* $\mathrm{v}_0 \in \mathrm{H}_0$*, there exists a subset* $\Gamma(\mathrm{v}_0)$ *of* $\mathbb{R}_+$*, of full Lebesgue measure, such that for each* $t \in \Gamma(\mathrm{v}_0)$, $U_t(\mathrm{v}_0) \in \mathrm{E}_0$*, and such that for* $T > 0$*, and every* $\mathrm{v}_0 \in \mathrm{H}_0$*, one has:*

$$\Big(\int_0^T |U_t \mathrm{v}_0|^p_{\mathrm{E}_0} dt\Big)^{\frac{1}{p}} \leq \tilde{C}_p(T)|\mathrm{v}_0|_{\mathrm{H}_0}. \tag{8.7}$$

Given $f \in L^1(0,T;H^0)$, set $U * f = \int_0^\cdot U_{\cdot-r} f(r)\,dr$; then we have the following inhomogeneous Strichartz inequalities.

Lemma 8.7. *Assume that Assumption 8.6 holds with* $p \in (2,\infty)$*. Then for* $T > 0$ *we have*

$$\Big|U * f\Big|_{L^p(0,T;\mathrm{E}_0)} \leq C\tilde{C}_p(T)\,|f|_{L^1(0,T;\mathrm{H}_0)}, \quad f \in L^1(0,T;\mathrm{H}_0), \tag{8.8}$$

$$\Big|U * f\Big|_{C([0,T];\mathrm{H}_0)} \leq C|f|_{L^1(0,T;\mathrm{H}_0)}, \quad f \in L^1(0,T;\mathrm{H}_0). \tag{8.9}$$

where $C = \sup_{t\in[-T,T]} |U_t|_{\mathcal{L}(H_0)} \in (0,\infty)$.

In the next lemma we show that once Assumption 8.6 holds for a certain set of objects it holds for a much larger class of sets of objects.

Lemma 1. *Assume that the spaces* H_0 *and* E_0*, and the operator* A *satisfy Assumption8.6 with a number* p*. Assume that* $\hat{s} \geq 0$ *and put* $\mathrm{H} = D((-A)^{\frac{\hat{s}}{2}}))$*, a subspace of* H_0*. Assume also that* $\mathrm{E} \subset E_0$ *is a separable Banach space such that* $\mathrm{E} \supset D((-\tilde{A})^{\frac{\hat{s}}{2}})$*. Then the spaces* H *and* E*, and the restriction of the operator* A *to* H *and* E *satisfy Assumption 8.6 with the same number* p*.*

This yields the following:

Corollary 1. *In the framework of Lemma 1, there exists a non decreasing function* $C_p : (0,\infty) \to (0,\infty)$ *such that* $C_p(0^+) = 0$*, and such that for every* $u_0 \in L^p(\Omega, \mathrm{H})$ *and* $T > 0$ *the trajectories of the process* $(U_t u_0, t \in [0,T])$ *belong a.s. to* $C([0,T];\mathrm{E}) \cap L^p(0,T;\mathrm{H})$*; moreover*

$$\mathbb{E}\Big(\int_0^T |U_t u_0|_{\mathrm{E}}^p\,dt + \sup_{t\in[0,T]} |U_t u_0|_{\mathrm{H}}^p\Big) \leq C(1 + C_p^p(T))\mathbb{E}|u_0|_{\mathrm{H}}^p. \tag{8.10}$$

We make the following assumption.

Assumption 8.8. *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*, where* $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$*, is a filtered probability space satisfying the usual assumptions. We assume that* $W = (W(t), t \geq 0)$ *is an* K*-cylindrical Wiener process on some real separable Hilbert space* K*; see Definition* (4.1) *(and/or Defintion 4.1 in* [15]*).*

Next let us recall Corollary 4.15

Corollary 8.9. *Let* E *be a martingale type* 2 *Banach space and* $p \in [2,\infty)$*. Then there exists a constant* $\hat{B}_p(E)$ *depending on* E *such that for every* $T \in (0,\infty]$ *and every* $L^p(0,T;E)$*-valued progressively measurable process* $(\Xi_s,\ s \in [0,T))$

$$\mathbb{E}\Big|\int_0^T \Xi_s\,dW(s)\Big|_{L^p(0,T;E)}^p \leq \hat{B}_p(E)\,\mathbb{E}\Big(\int_0^T \|\Xi_s\|_{R(\mathrm{K},L^p(0,T;E))}^2\,ds\Big)^{p/2}. \tag{8.11}$$

Moreover, for any $T > 0$*, the above inequality* (8.11) *holds true also for the space* $L^p(0,T;E)$ *and the integral over interval* $(0,T)$ *with the same constant* $\hat{B}_p(E)$*.*

Using the above result with a progressively measurable $R(\mathrm{K}, L^p(0,T;E_0))$-valued process $(\Xi_r)_{r\in[0,T]}$ defined by

$$\Xi_r := \{[0,T] \ni t \mapsto 1_{[r,T]}(t) U_{t-r}\xi(r)\}, \quad r \in [0,T]. \tag{8.12}$$

we can prove the following stochastic Strichartz inequality:

Theorem 8.10. *Assume that Assumption 8.6 is satisfied and* E_0 *is a martingale type* 2 *Banach space. For each* $T > 0$ *there exist a constant* $\hat{C}_p(T) > 0$ *such that for every predictable process* $\xi \in \mathcal{M}_{loc}^p([0,\infty), \mathrm{R}(\mathrm{K},\mathrm{H}_0))$ *and every accessible stopping time* τ *satisfying* $\tau \leq T$ *and* $\mathbb{E}(\int_0^\tau \|\xi(t)\|_{R(K,\mathrm{H}_0)}^2\,dt)^{\frac{p}{2}} < \infty$*, one has*

$$\mathbb{E}\int_0^T |[J_{[0,\tau)}\xi](t)|_{\mathrm{E}_0}^p\,dt \leq \hat{C}_p(T)\mathbb{E}\Big(\int_0^\tau \|\xi(t)\|_{\mathrm{R}(\mathrm{K},\mathrm{H}_0)}^2\,dt\Big)^{p/2}, \tag{8.13}$$

where

$$[J_{[0,\tau)}\xi](t) = \int_0^t 1_{[0,\tau)}(r)U_{t-r}\xi(r)\,dW(r), \quad t \geq 0. \tag{8.14}$$

Next we formulate a result which is related to inequality (8.13) in a way similar to that the inhomogeneous inequality (8.9) is related to homogeneous inequality (8.8).

Proposition 8.11. *Asumme that the assumptions of Theorem 8.10 are satisfied; then there exists a constant $C_p > 0$ such that for ξ and τ as in Theorem 8.10 we have:*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \Big|[J_{[0,\tau)}\xi](t)\Big|^p_{\mathrm{H}_0}\Big) \leq C_p\mathbb{E}\Big[\int_0^\tau \|\xi(t)\|^2_{R(K,\mathrm{H}_0)}dt\Big]^{\frac{p}{2}}. \tag{8.15}$$

Thus we have the following version of Theorem 8.10 using Lemma 1.

Corollary 8.12. *Assume that the assumptions of Lemma 1 are satisfied. Then for each $T > 0$ there exists a constant $\hat{C}_p(T)$ such that:*

(i) $\lim_{T\to 0} \hat{C}_p(T) = 0$,

(ii) *For every finite accessible $\mathbb{F}$-stopping time T_0, every $\mathbb{F}^{T_0}$ stopping time τ bounded by T and every process $\xi \in M^p_{loc}([0,\infty), \mathbb{F}^{T_0}, \mathrm{R(K,H)})$ such that $\mathbb{E}(\int_0^\tau \|\xi(t)\|^2_{\mathrm{R}(\mathrm{K},H)}dt)^{p/2} < \infty$ one has*

$$\mathbb{E}\int_0^T |J^{T_0}_{[0,\tau)}\xi(t)|^p_E\,dt \leq \hat{C}_p(T)\mathbb{E}\Big(\int_0^\tau \|\xi(t)\|^2_{\mathrm{R}(\mathrm{K},H)}\,dt\Big)^{\frac{p}{2}}. \tag{8.16}$$

where $J^{T_0}_{[0,\tau)}$ is a generalization of the process $J_{[0,\tau)}\xi$ defined in (8.14) *as the starting time is equal to the stopping time T, i.e.*

$$[J^{T_0}_{[0,\tau)}\xi](t) = \int_0^t 1_{[0,\tau)}(r)U_{t-r}\xi(r)\,dW^{T_0}(r), \quad t \geq 0, \tag{8.17}$$

8.1.1. *Examples of the deterministic and the stochastic Strichartz estimates.* Let M be a compact Riemannian manifold M of dimension $d \geq 2$. According to Burq et all [16], Assumption 8.6 is satisfied by the Hilbert space $\mathcal{H}_0 = L^2(M)$, the C_0-group of unitary operators $(U_t, t \in \mathbb{R})$ with infinitesimal generator iA, where $A := \Delta$ is the Laplace-Beltrami operator on M, and the spaces $\mathrm{H}_0 = H^{\frac{1}{p},2}(M)$ and $\mathrm{E}_0 = L^q(M)$, provided the parameters, $p \in [2,\infty)$ and $q \in (2,\infty)$ satisfy the so called scaling admissible condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \tag{8.18}$$

Indeed, on $H_0 = H^{\frac{1}{p},2}(M)$ we may consider either the norm $\|.\|_{H^{1/p,2}(M)}$ or, since $H_0 = D((-\Delta)^{1/(2p)})$ the equivalent norm $|\Delta^{1/(2p)}.|_{L^2(M)}$ for which $(U_t, t \in \mathbb{R})$ is a group of isometries.

It is well known that when M is replaced by $\mathbb{R}^d$, then Assumption 8.6 is satisfied by the C_0 unitary group generated by the operator $i\Delta$, and the spaces $\mathcal{H}_0 = \mathrm{H}_0 = L^2(\mathbb{R}^d)$ and $\mathrm{E}_0 = L^q(\mathbb{R}^d)$ provided $p \in [2,\infty)$ and $q \in (2,\infty)$ satisfy the scaling

admissible condition (8.18). In this setting, the identity (8.18) is optimal with these spaces for (8.7) to hold true.
It is shown in [16, Theorem 4] that when $M = S^2$ is the two-dimensional sphere, then Assumption 8.6 is satisfied by the C_0-group of unitary operators generated by the operator $i\Delta$ with the following choice parameters: $p = 4$, $\mathrm{E}_0 = L^4(M)$ and $\mathrm{H}_0 = H^{s,2}(M)$ for $s > s_0(2) = \frac{1}{8}$, which proves that (8.8) is not optimal for $\mathrm{H}_0 = H^{\frac{1}{4},2}(M)$ and $\mathrm{E}_0 = L^4(M)$. Note also that (8.8) does not hold when $s < s_0(2)$. The following result proves that on compact manifolds, the homogenous Strichartz inequality (8.7) and Lemma 8.7 hold for the following spaces: $\mathrm{H} = H^{\sigma+\frac{1}{p},2}(M)$ and $\mathrm{E} = W^{\sigma,q}(M)$, for $\sigma \geq 0$.

Proposition 8.13. *Let M be compact Riemanian manifold, $(U(t) = e^{it\Delta}, t \in \mathbb{R})$, (p, q) satisfy the scaling admissible condition (8.18). Then for each $\sigma \geq 0$ and $T > 0$ there exists a constant $\bar{C}_q(T) > 0$ such that $\lim_{T\searrow 0} \bar{C}_q(T) = 0$ and*

(i) *For every $v_0 \in H^{\sigma+\frac{1}{p},2}(M)$,*

$$\Big(\int_0^T \|U(t)v_0\|^p_{W^{\sigma,q}} dt\Big)^{1/p} \leq \bar{C}_q(T)\|v_0\|_{H^{\sigma+\frac{1}{p},2}}. \tag{8.19}$$

(ii) *For every $g \in L^1(0, T; H^{\sigma+\frac{1}{p},2}(M))$,*

$$\Big(\int_0^T \|(U * g)(t)\|^p_{W^{\sigma,q}} dt\Big)^{1/p} \leq \bar{C}_q(T) \int_0^T \|g(t)\|_{H^{\sigma+\frac{1}{p},2}} dt. \tag{8.20}$$

Proof. This result follows Lemma 8.7 and Lemma 1. Set $\mathcal{H}_0 = L^2(M)$. First let us notice that for $\sigma = 0$, $\mathrm{H}_0 = H^{\frac{1}{p},2}(M)$ and $\mathrm{E}_0 = L^q(M)$ the above inequalities are satisfied by Lemma 8.7 since in view of [16] Assumption 8.6 is satisfied for this choice of spaces. Let us denote by A_r the version of the operator A on the space $L^r(M)$, $r \in [1, \infty)$. Then the space $\mathrm{H} = H^{\frac{1}{p}+\sigma,2}(M)$ is equal to $D(A_2^{\frac{1}{2p}+\frac{\sigma}{2}})$. Moreover, $H^{\sigma,q}(M) = D(A_q^{\frac{\sigma}{2}})$; since $q \in (2, \infty)$ we have $H^{\sigma,q}(M) \subset W^{\sigma,q}(M) =: \mathrm{E}$. The proof is complete. □

Similarly, Corollary 1 has the following particular formulation.

Corollary 8.14. *In the framework of Proposition 8.13, if $\mathrm{H} = H^{\sigma+\frac{1}{p},2}(M)$ and either $\mathrm{E} = H^{\sigma,q}(M)$ or $\mathrm{E} = W^{\sigma,q}(M)$, then for every $u_0 \in L^p(\Omega, \mathrm{H})$ and every $T > 0$ the trajectories of the process $(U_t u_0, t \in [0, T])$, belong a.s. to $C([0, T]; \mathrm{E}) \cap L^p(0, T; \mathrm{H})$ and moreover*

$$\mathbb{E}\Big(\int_0^T |U_t u_0|^p_{\mathrm{E}} dt + \sup_{t\in[0,T]} |U_t u_0|^p_{\mathrm{H}}\Big) \quad \leq \quad (1 + \bar{C}^q_p(T))\mathbb{E}|u_0|^p_{\mathrm{H}} \tag{8.21}$$

Finally note that Corollary 8.12 holds for $\mathrm{H} = H^{\sigma+\frac{1}{p},2}(M)$ and $\mathrm{E} = W^{\sigma,q}(M)$ for any $\sigma \geq 0$, and for $p \in [2, \infty)$, $q \in (2, \infty)$ satisfying the scaling admissible condition $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$.

8.2. **Stochastic NSEs: abstract local existence result.** The aim of this section is to prove an abstract local existence result that will be used subsequently to prove the local existence for certain nonlinear Schrödinger equations. This section is divided into five subsections.

8.3. **Asumptions and truncated equation.** We begin with a description of the main assumptions. The first one of them is just Assumption 8.6.

Assumption 8.15.(i) *Assume that the spaces* H_0 *and* E_0 *and the operator* A *satisfy Assumption 8.6 with some number* $p \in (2, \infty)$. *Let* $\hat{s} \geq 0$ *and put* $\mathrm{H} = D((-A)^{\frac{\hat{s}}{2}}) \subset \mathrm{H}_0$. *Assume also that* $\mathrm{E} \subset E_0$ *is a separable Banach space such that* $\mathrm{E} \supset D((-\tilde{A})^{\frac{\hat{s}}{2}})$.

(ii) *Assume that* F *is a locally Lipschitz map from* $\mathrm{H} \cap \mathrm{E}$ *to* H *in the following sense. There exists positive constants* C *and* $\beta \in [1, p)$ *such that for all* $u, v \in \mathrm{H} \cap \mathrm{E}$

$$|F(u)|_{\mathrm{H}} \leq C\,[(1 + |u|_{\mathrm{E}}^{\beta}) + (1 + |u|_{\mathrm{E}}^{\beta-1})|u|_{\mathrm{H}}], \tag{8.22}$$

$$\begin{aligned} \big|F(u) - F(v)\big|_{\mathrm{H}} \leq C\,[1 + |u|_{\mathrm{E}}^{(\beta-2)^+} + |v|_{\mathrm{E}}^{(\beta-2)^+}][1 + |u|_{\mathrm{H}} + |v|_{\mathrm{H}}]|u - v|_{\mathrm{E}} \\ + C\,[1 + |u|_{\mathrm{E}}^{\beta-1} + |v|_{\mathrm{E}}^{\beta-1}]|u - v|_{\mathrm{H}}. \end{aligned} \tag{8.23}$$

(iii) *Assume that* G *is a locally Lipschitz map from* $\mathrm{H} \cap \mathrm{E}$ *to* $R(K, \mathrm{H})$ *in the following sense. There exist positive constants* C *and* $a \in [1, p/2)$ *such that for all* $u, v \in \mathrm{H} \cap \mathrm{E}$

$$|G(u)|_{R(K,\mathrm{H})} \leq C\,[(1 + |u|_{\mathrm{E}}^{a}) + (1 + |u|_{\mathrm{E}}^{a-1})|u|_{\mathrm{H}}], \tag{8.24}$$

$$\begin{aligned} |G(u) - G(v)|_{R(K,\mathrm{H})} \leq C\,(1 + |u|_{\mathrm{E}}^{a-1} + |v|_{\mathrm{E}}^{a-1})|u - v|_{\mathrm{H}} \\ + C\,(1 + |u|_{\mathrm{E}}^{(a-2)^+} + |v|_{\mathrm{E}}^{(a-2)^+})(1 + |u|_{\mathrm{H}} + |v|_{\mathrm{H}})|u - v|_{\mathrm{E}}. \end{aligned} \tag{8.25}$$

We use the convention $x^0 = 1$. Lemma 1 implies that the spaces H and E satisfy the Assumption 8.6. Although the above growth and local Lipschitz continuity conditions are a bit unusual one can easily see that, as in the more typical situations, the inequality (8.23) implies (8.22) and the inequality (8.25) implies (8.24) if $\beta, a \geq 2$.

In this section we will consider the following stochastic Itô nonlinear Schrödinger Equation of the following form:

$$idu(t) + Au(t)\,dt = F(u)\,dt + G(u)\,dW(t), \quad u(0) = u_0, \tag{8.26}$$

where the initial data u_0 belongs to the Hilbert space H. For $d \geq c \geq 0$, let us denote

$$Y_{[c,d]} := C([c, d]; \mathrm{H}) \cap L^p(c, d; \mathrm{E}), \tag{8.27}$$

Obviously, $Y_{[c,d]}$ is a Banach space with norm defined by:

$$|u|_{Y_{[c,d]}}^p := \sup_{r \in [c,d]} |u(r)|_{\mathrm{H}}^p + \int_c^d |u(r)|_{\mathrm{E}}^p\,ds.$$

Note that the $(Y_{[c,t]})_{t \geq c}$ is an non increasing family of Banach spaces. More precisely, if $t > \tau > c$ and $u \in Y_{[c,t]}$, then $u_{|[c,\tau]} \in Y_{[c,\tau]}$ and $|u_{|[c,\tau]}|_{Y_{[c,\tau]}} \leq |u|_{Y_{[c,t]}}$. To ease notation, we will simply write $Y_t = Y_{[0,t]}$.

Let $\mathbb{M}^p(Y_{T_1},\mathbb{F}^{T_0}) := \mathbb{M}^p(Y_{[0,T_1]},\mathbb{F}^{T_0})$ denote the Banach space of continuous H-valued $\mathbb{F}^{T_0}$-adapted local processes $(X_t, t\in[0,T_1])$ which satisfy

$$\|X\|^p_{\mathbb{M}^p(Y_{T_1},\mathbb{F}^{T_0})} = \mathbb{E}\Big(\sup_{r\in[0,T_1]}|X(t)|^p_{\mathrm{H}} + \int_0^{T_1}|X(r)|^p_{\mathrm{E}}dr\Big) < \infty. \tag{8.28}$$

Similarly, let $\mathbb{M}^p_{\mathrm{loc}}(Y_{[0,T_1)},\mathbb{F}^{T_0})$ denote the set of all H-valued $\mathbb{F}^{T_0}$-adapted and continuous local processes $(X_t, t\in[0,T_1))$ such that $X\in\mathbb{M}^p(Y_{\tau_n},\mathbb{F}^{T_0})$ for any sequence of $\mathbb{F}^{T_0}$ stopping times (τ_n) approximating T_1.
Now we will introduce definitions of local and maximal local solutions; they are modifications of definitions used earlier, such as in [6], [7] and [10].

Definition 8.16. Assume that T_0 is a finite accessible $\mathbb{F}$-stopping time and u_0 is a H-valued $\mathcal{F}_{T_0}$-measurable random variable. A **local mild solution** to problem (8.26) with initial condition u_0 at time T_0 is a process u defined as $u(T_0+t)=X(t)$, $t\in[0,T_1)$, where
(i) T_1 is an accessible $\mathbb{F}^{T_0}$ stopping time,
(ii) $X=(X(t), t\in[0,T_1))$ belongs to $\mathbb{M}^p_{\mathrm{loc}}(Y_{[0,T_1)},\mathbb{F}^{T_0})$,
(iii) for some approximating sequence (τ_n) of $\mathbb{F}^{T_0}$ stopping times for T_1, one has,

$$X(t\wedge\tau_n) = U_{t\wedge\tau_n}u_0 + \int_0^{t\wedge\tau_n} U_{t\wedge\tau_n-r}F(X(r))\,dr + I_{\tau_n}(G(X))(t), \tag{8.29}$$

for every every $n=1,2,\cdots$ and $t\geq 0$, where $I_{\tau_n}(G(X))$ is the process defined by

$$I_{\tau_n}(G(X))(t) = \int_0^\infty 1_{[0,t\wedge\tau_n]}(r)U_{t-r}G(X(r))\,dW^{T_0}(r). \tag{8.30}$$

A local mild solution $u=(u(T_0+t)\,,\,0\leq t<T_1)$ to problem (8.26) is **pathwise unique** if for any other local mild solution $\tilde{u}=(\tilde{u}(T_0+t)\,,\,0\leq t<\tilde{T}_1)$ for this problem, $u(T_0+t,\omega)=\tilde{u}(T_0+t,\omega)$ for almost every $(t,\omega)\in[0,T_1\wedge\tilde{T}_1)\times\Omega$.
A local mild solution $u=(u(T_0+t)\,,\,t\in[0,T_1))$ is called **maximal** if for any other local mild solution $\tilde{u}=(\tilde{u}(T_0+t)\,,\,t\in[0,\tilde{T}_1))$ satisfying $\tilde{T}_1\geq T_1$ a.s. and $\tilde{u}|_{[T_0,T_0+T_1)\times\Omega}\sim u$, one has $T_1=\tilde{T}_1$ a.s. The $\mathbb{F}$-stopping time T_0+T_1 will be called the life span of the maximal local mild solution u. Furthermore, a maximal local mild solution $(u(T_0+t),\ t\in[0,T_1))$ is called global if its lifespan is equal to ∞ a.s., i.e. $T_1=\infty$ a.s.

The existence and uniqueness of a local maximal solution to problem (8.26) will be proved in section 8.4. We at first prove the existence and the uniqueness of the solution when its norm is truncated. Thus let $\theta:\mathbb{R}_+\to[0,1]$ be a C_0^∞ non increasing function such that

$$\inf_{x\in\mathbb{R}_+}\theta'(x)\geq -1, \quad \theta(x)=1 \text{ iff } x\in[0,1] \quad \text{and } \theta(x)=0 \text{ iff } x\in[2,\infty), \tag{8.31}$$

and for $n\geq 1$ set $\theta_n(\cdot)=\theta(\frac{\cdot}{n})$. Let us fix some finite accessible $\mathbb{F}$-stopping time T_0, some constant $T>0$ and some accessible $\mathbb{F}^{T_0}$-stopping time T_1 such that $T_1\leq T$. The rest of this section is devoted to prove existence and uniqueness of the solution

X^n to the following evolution equation for $t \in [0, T_1]$:

$$X^n(t) = U_t u^n(T_0) + \int_0^t U_{t-r}[\theta_n(|X^n|_{Y_r}) F(X^n(r))]\, dr$$
$$+ \int_0^t U_{t-r}[\theta_n(|X^n|_{Y_r}) G(X^n(r))]\, dW^{T_0}(r). \tag{8.32}$$

8.3.1. *Existence and uniqueness of a global solutions to approximating equations.* A direct consequence of all the results proved and the Banach-Cacciopoli Fixed Point Theorem is presented below.

Theorem 8.17. *Assume that the Assumptions 8.8, 8.6 and 8.15 are satisfied with $\beta \in [2, p)$ and $a \in [1, \frac{p}{2})$. Assume also that* E *is a martingale type* 2 *Banach space. Let T_0 be a finite and accessible $\mathbb{F}$-stopping time T_0 and $u(T_0) \in L^p(\Omega, \mathcal{F}_{T_0}, \mathrm{H})$. Then for every positive integer n, there exists a unique process $u^n = (u^n(t), t \in [T_0, \infty))$, such that $u^n(t + T_0) = X^n(t)$ for $t \geq 0$ and for every $T > 0$, X^n belongs to the space $\mathbb{M}^p(Y_T, \mathbb{F}^{T_0})$ and $X^n = \mathcal{U}_{[0,T]}(u(T_0)) + \Phi^n_T(X^n) + \Psi^{T_0,n}_T(X^n)$. Moreover, given any $T > 0$ the process X^n is the unique solution to the evolution equation* (8.32) *on the time interval* $[0, T]$*. Moreover, if a local process $v = (v(T_0 + t) = \tilde{X}(t), t \in [0, \tau))$ is a local solution to* (8.32)*, then the processes X^n and $\bar{X} := \tilde{X}\big|_{[0,\tau)\times\Omega}$ are equivalent.*

8.4. **Existence of a local maximal mild solution to problem** (8.26)**.** Our first aim is to prove the following result about the existence and uniqueness of a local maximal solution to problem (8.26). Let T_0 be a finite accessible $\mathbb{F}$-stopping time and suppose that the assumption 8.15 on F and G is satisfied. Let $u(T_0)$ be a p-integrable H-valued $\mathcal{F}_{T_0}$ random variable. In the previous section (see Theorem 8.17) we proved for every $n \in \mathbb{N}$ the existence of a unique solution $u^n(T_0+.) = X^n(.)$ on $[0, \infty)$ to the problem (8.32). Let τ_n and $\hat{\tau}_n$ denote the $\mathbb{F}^{T_0}$ stopping times defined by

$$\tau_n = \inf\{t > 0 : |X^n|_{Y_t} \geq n\} \wedge n, \qquad \hat{\tau}_n = \inf\{t > 0 : |X^n|_{Y_t} \geq 2n\} \wedge n, \tag{8.33}$$

The following result establishes the existence and uniqueness of a local solution to (8.32).

Proposition 8.18. *Let $(X^n(t), t \geq 0)$ be the process introduced in Theorem 8.17. Then the process $(X^n(t), t < \tau_n)$ is a local mild solution to problem* (8.26) *with the filtration $\mathbb{F}^{T_0}$ and the Brownian Motion W^{T_0}.*

In the following Theorem we will prove the existence of a unique local mild solution to problem (8.26). An important feature of this result is that we can estimate from below the length of the existence time interval with a lower bound, which depends on the p-th moment of the H-norm of the initial data, on a "large" subset of Ω whose probability does not depend on this moment. This property is used later on in proving that the H-norm of solution converges to ∞ as the time converges to the lifespan, provided it is finite.

Theorem 8.19. *Let us assume that Assumptions 8.6 and 8.15 be satisfied. Then for every $\mathcal{F}_{T_0}$-measurable* H*-valued p-integrable random variable $u(T_0)$ there exits a local process $X = (X(t), t \in [0, T_1))$ which is the unique local mild solution to the problem* (8.26) *with the filtration $\mathbb{F}^{T_0}$ and the Brownian W^{T_0}. Moreover, given $R > 0$ and $\varepsilon > 0$ there exists $\tau(\varepsilon, R) > 0$, such that for every $\mathcal{F}_{T_0}$-measurable* H*-valued random variable $u(T_0)$ satisfying $\mathbb{E}|u(T_0)|^p_{\mathrm{H}} \leq R^p$, one has $\mathbb{P}(T_1 \geq \tau(\varepsilon, R)) \geq 1 - \varepsilon$.*

Assume that $T_0 = 0$, $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathrm{H})$ and let τ_n be defined by (8.33). Set

$$\tau_\infty(\omega) := \lim_{n \nearrow \infty} \tau_n(\omega), \;\; \omega \in \Omega.$$

Then $U^n = X^n$ and by local uniqueness we infer that that the following identity uniquely defines a local process $(u(t)\, ,\, t < \tau_\infty)$ as follows:

$$u(t, \omega) := u^n(t, \omega), \;\; \text{if } t < \tau_n(\omega), \;\; \omega \in \Omega. \tag{8.34}$$

We can now prove the existence and uniqueness of a maximal solution to our abstract evolution equation (8.26); this is the main result of this section.

Theorem 1. Assume that the Assumptions 8.8, 8.6 and 8.15 are satisfied with $2 \leq \beta < p$ and $a \in [1, \frac{p}{2})$. Assume also that E is a martingale type 2 Banach space. Then for every finite and accessible $\mathbb{F}$-stopping time T_0 and every $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathrm{H})$, the process $u = (u(t)\, ,\, t < \tau_\infty)$ defined by (8.34) is the unique local maximal solution to problem (8.26). Moreover, $\mathbb{P}(\{\tau_\infty < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_{\mathrm{H}} < \infty\}) = 0$ and on $\{\tau_\infty < \infty\}$, $\limsup_{t \to \tau_\infty} |u(t)|_{\mathrm{H}} = +\infty$ a.s.

8.5. **Abstract Stochastic NLS in the Stratonovich form.** Multiplying equation (8.26) by $-i$, we obtain the following form of it:

$$du(t) = i\left[Au(t) - F(u)\right]dt + (-i)\, G(u)\, dW(t), \;\; t \geq 0.$$

Now we suppose that the stochastic term is in the Stratonovich form, i.e. formally

$$du(t) = i\left[Au(t) - F(u)\right]dt + (-i)\, G(u) \,\circ\, dW(t), \;\; t \geq 0. \tag{8.35}$$

Below we will present a rigorous approach to equation (8.35). We assume that the assumptions of Theorem 1 are satisfied. In order for this problem to make sense, we need to make stronger assumption on the map G. To be precise, we require the following assumptions.

Assumption 8.20. *The Hilbert space* K *is such that*

$$\mathrm{K} \subset \mathcal{R} := \mathrm{H} \cap \mathrm{E},$$

and the natural embedding $\Lambda : \mathrm{K} \hookrightarrow \mathcal{R}$ is γ-radonifying.

Assumption 8.21. *The map $G : \mathcal{R} \to \mathcal{L}(\mathcal{R}, \mathcal{R})$ is of real C^1-class.*

Note that the above Assumptions 8.20 and 8.21 imply that the naturally induced map

$$G : \mathcal{R} \ni u \mapsto G(u) \circ \Lambda \in \mathrm{R}(\mathrm{K}, \mathcal{R}),$$

which can be identified with the original map $G : \mathcal{R} \to \mathcal{L}(\mathcal{R}, \mathcal{R})$, is of real C^1-class and satisfies

$$|G(u)|_{R(\mathrm{K},\mathcal{R})} \leq |G(u)|_{\mathcal{L}(\mathcal{R},\mathcal{R})} \, |\Lambda|_{R(K,\mathcal{R})}.$$

Furthermore, for $u \in \mathcal{R}$, the Fréchet derivative $G'(u) = d_u G \in \mathcal{L}(\mathcal{R}, \mathcal{L}(\mathcal{R}, \mathcal{R}))$ is $\mathbb{R}$-linear. By the Kwapień-Szymański Theorem [31] we can assume that an ONB $\{e_j\}_{j\geq 1}$ of K can be chosen (and fixed for the remainder of the article) in such a way that

$$\sum_{j\geq 1} |\Lambda e_j|^2_{\mathcal{R}} < \infty. \tag{8.36}$$

Let us recall that for a bilinear $\phi \in \mathbb{L}_2(\mathcal{R};\mathcal{R})$, we put

$$\mathrm{tr}_{\mathrm{K}}(\phi) := \sum_{j\geq 1} \phi(\Lambda e_j, \Lambda e_j) \in \mathcal{R}. \tag{8.37}$$

Following [7] we can define the Stratonovich differential $-iG(u)\circ dW(t)$ as follows:

$$\begin{aligned} -iG(u) \circ dW(t) &= -iG(u)\, dW(t) + \frac{1}{2}\mathrm{tr}_{\mathrm{K}}(- iG'(u))(- iG(u))\, dt \\ &= -iG(u)\, dW(t) + \frac{1}{2}\mathrm{tr}_{\mathrm{K}}(iG'(u))(iG(u))\, dt. \end{aligned} \tag{8.38}$$

If for $u \in \mathcal{R}$, we denote by $\mathcal{M}(u) = (iG'(u))(iG(u))$ the element of $\mathcal{L}(\mathcal{R}, \mathcal{L}(\mathcal{R}, \mathcal{R})) \equiv \mathbb{L}_2(\mathcal{R};\mathcal{R})$ defined by
(8.39)

$$\mathcal{M}(u)(h_1, h_2) = (iG'(u))(iG(u))(h_1, h_2) = (iG'(u)(iG(u)h_1)h_2, \quad h_1, h_2 \in \mathcal{R},$$

then equation (8.35) can be reformulated in the following way:

$$du = \Big[iAu - iF(u) + \frac{1}{2}\mathrm{tr}_{\mathrm{K}}(\mathcal{M}(u))\Big]\, dt + iG(u)dW(t). \tag{8.40}$$

Given real-valued maps ϕ, ψ we write $\phi \lesssim \psi$ if there exists a constant c such that $\phi \leq c\psi$. We write $\phi \eqsim \psi$ to express that $\phi \lesssim \psi$ and $\psi \lesssim \phi$.
Let us recall that although $\mathcal{R}$ is a complex Banach space, below we will treat it as a real Banach space. The following result states the equivalence of the $\mathbb{L}_2(\mathcal{R};\mathcal{R})$ norm of $\mathcal{M}(u)$ and of $G'(u)G(u)$. Its proof, which is is straightforward, is omitted.

Lemma 8.22. *Assume that the multiplication by i is a bounded real linear map in the real Banach space $\mathcal{R}$. Then we have, for all $u, v \in \mathcal{R}$,*

$$\begin{aligned} |\mathcal{M}(u)|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})} &\eqsim |G'(u)(G(u))|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})}, \\ |\mathcal{M}(u) - \mathcal{M}(v)|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})} &\eqsim |G'(u)(G(u)) - G'(v)(G(v))|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})}, \end{aligned}$$

where $G'(u)(G(u))(h_1, h_2) = G'(u)(G(u)h_1)h_2$ for $h_1, h_2 \in \mathcal{R}$.

Since $\mathcal{R} \hookrightarrow$ H continuously and the trace

$$\mathrm{tr}_{\mathrm{K}} : \mathbb{L}_2(\mathcal{R};\mathcal{R}) \ni \phi \mapsto \mathrm{tr}_{\mathrm{K}}(\phi) \in \mathcal{R}$$

is a linear and bounded map, we can find $C > 0$ such that

$$\Big|\mathrm{tr}_{\mathrm{K}}\mathcal{M}(u) - \mathrm{tr}_{\mathrm{K}}\mathcal{M}(v)\Big|_H \leq C|\mathcal{M}(u) - \mathcal{M}(v)|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})}, \quad u, v \in \mathcal{R}.$$

Definition 8.23. We say that a process u is a local (resp. local maximal, global) solution to equation (8.35) if and only if it is a local (resp. local maximal, global) solution to the Itô equation (8.40), that is to problem (8.26), with the map F being replaced by $F_1 = F + \frac{i}{2}\mathrm{tr}_K\big[\mathcal{M}\big]$.

We now state the stronger version of Assumption 8.15(iii).

Assumption 8.24. *Let $p \in (2,\infty)$, $a \in [1, \frac{p}{2})$ and $\gamma \in [1,p)$. The map $G : \mathcal{R} \to \mathcal{L}(\mathcal{R},\mathcal{R})$ is of class C^1 and such that for some positive constant C, and all $u, v \in \mathcal{R}$, with $\mathcal{M}$ having been defined in* (8.39)

$$|G(u)|_{\mathcal{L}(\mathcal{R},\mathrm{H})} \le C\Big[1 + |u|_{\mathrm{E}}^{a} + (1 + |u|_{\mathrm{E}}^{a-1})|u|_{\mathrm{H}}\Big] \tag{8.41}$$

$$\begin{aligned}|G(u) - G(v)|_{\mathcal{L}(\mathcal{R},\mathrm{H})} &\le C(1 + |u|_{\mathrm{E}}^{a-1} + |v|_{\mathrm{E}}^{a-1})|u - v|_{\mathrm{H}}\\ &\quad + C(1 + |u|_{\mathrm{E}}^{(a-2)^+} + |v|_{\mathrm{E}}^{(a-2)^+})(1 + |u|_{\mathrm{H}} + |v|_{\mathrm{H}})|u - v|_{\mathrm{E}},\end{aligned} \tag{8.42}$$

$$\Big|\mathcal{M}(u)\Big|_{\mathbb{L}_2(\mathcal{R};\mathrm{H})} \le C\Big[1 + |u|_{\mathrm{E}}^{\gamma} + (1 + |u|_{\mathrm{E}}^{\gamma-1})|u|_{\mathrm{H}}\Big], \tag{8.43}$$

$$\begin{aligned}\Big|\mathcal{M}(u) - \mathcal{M}(v)\Big|_{\mathbb{L}_2(\mathcal{R};\mathrm{H})} &\le C(1 + |u|_{\mathrm{E}}^{\gamma-1} + |v|_{\mathrm{E}}^{\gamma-1})|u - v|_{\mathrm{H}}\\ &\quad + \ C(1 + |u|_{\mathrm{E}}^{(\gamma-2)^+} + |v|_{\mathrm{E}}^{(\gamma-2)^+})(1 + |u|_{\mathrm{H}} + |v|_{\mathrm{H}})|u - v|_{\mathrm{E}}.\end{aligned} \tag{8.44}$$

Theorem about the existence of local maximal solution to an abstract stochastic NLS immediately yields the following result.

Theorem 8.25. *Assume that the Assumptions 8.6 and 8.15 are satisfied with $2 \le \beta < p$ and $a \in [1, \frac{p}{2})$ and also Assumptions 8.20, 8.21 and 8.24 are satisfied. Assume also that* E *is a martingale type* 2 *Banach space and $W = (W_t)_{t\ge 0}$ is an $\mathcal{R} = \mathrm{H}\cap\mathrm{E}$-valued Wiener process defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual assumptions. Then for every $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathrm{H})$ there exists a unique local process $u = (u(t)\,,\, t < \tau_\infty)$ which is the local maximal solution to problem (8.35). Moreover, $\mathbb{P}(\{\tau_\infty < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_{\mathrm{H}} < \infty\}) = 0$ and on $\{\tau_\infty < \infty\}$, we have $\limsup_{t\to\tau_\infty} |u(t)|_{\mathrm{H}} = +\infty$ a.s.*

8.6. **Stochastic NSEs: the local existence.** In this section we will formulate results about the existence and the uniqueness of solutions to the stochastic NLS equation, in the Itô (respectively Stratonovich) formulation. The former one will be based on an abstract result on the existence of local maximal solution and the latter on Theorem 8.25. For simplicity we formulate it for $d = 2$. One can also prove a similar result for $d > 2$ but since in the latter case we do not know whether the solution is global or blows up in finite time, we have decided to leave it out. Thus we assume that M is a 2-dimensional, compact riemannian manifold and Δ is the Laplace Beltrami operator on M. We assume that some numbers p, q satisfy the scaling admissible condition (with $d = 2$)

$$\frac{2}{p} + \frac{2}{q} = 1. \tag{8.45}$$

We choose $s \in (1-\frac{1}{p}, 1]$ and put $\hat{s} := s - \frac{1}{p}$. Since $s - \frac{1}{p} > \frac{d}{q}$, we infer that the Sobolev space $W^{\hat{s},q}(M)$ is embedded into the space $C(M)$ of continuous (and hence bounded) functions on M; the latter is a Banach space equipped with the L^∞-norm. In this section we let $\mathrm{H} = H^{s,2}(M) := H^{s,2}(M, \mathbb{C})$ and $\mathrm{E} = W^{\hat{s},q}(M) := W^{\hat{s},q}(M, \mathbb{C})$, where $\mathbb{C}$ is identified with $\mathbb{R}^2$. Finally, as in Assumption 8.20, we denote by $\mathcal{R}$ the following real Banach space

$$\mathcal{R} = H^{s,2}(M) \cap W^{\hat{s},q}(M) = \mathrm{H} \cap \mathrm{E}.$$

In order to study the diffusion operator G we need the following result which follows from Corollary 8.2 and Theorem 8.3.

Lemma 8.26. *Under the above assumptions the pointwise multiplication map*

$$\Pi : \mathcal{R} \times \mathcal{R} \ni (u, h) \mapsto uh \in \mathcal{R} \tag{8.46}$$

is bilinear and continuous. The same assertion holds for $\mathcal{R} = H^{s,2}(M) \cap L^\infty(M)$.

The following assumption will play an essential rôle in the next section as well as in the last part of Theorem 8.31, which is the main result of the current section.

Assumption 8.27. *There exists a function* $\tilde{g} : [0, +\infty) \to \mathbb{R}$ *of class* C^1 *such that*

$$g(z) = \tilde{g}(|z|^2)z, \; z \in \mathbb{C}. \tag{8.47}$$

We will consider the "generalized" Nemytski map $\tilde{G}$ associated with g (see [7]), that is with $\mathcal{R} = \mathrm{H} \cap \mathrm{E}$,

$$\tilde{G} : \mathcal{R} \ni u \mapsto \{h \mapsto \Pi(g(u), h)\} \in \mathcal{L}(\mathcal{R}, \mathcal{R}). \tag{8.48}$$

The aim of this section is to prove the existence and uniqueness of a maximal solution to the problem

$$idu(t) + \Delta u(t)\, dt = f(u)\, dt + g(u)\, dW(t), \quad u(0) = u_0, \tag{8.49}$$

or to its Stratonovich formulation

$$idu(t) + \Delta u(t)\, dt = f(u)\, dt + g(u) \circ dW(t), \quad u(0) = u_0. \tag{8.50}$$

By a solution of problem (8.49) (resp. (8.50)), we mean a solution to its abstract version (8.26) (resp. (8.40)) defined in terms of the Nemytski map $\tilde{G}$. As proved in the previous section, the Stratonovich formulation requires to identify $\mathcal{M}$ as the Nemytski map corresponding to function $z \mapsto (ig'(z))(ig(z))$. This will be a consequence of the following general result.

Lemma 8.28. *Assume that* $(H, |\cdot|)$ *is a real separable Hilbert space and that* $\mathcal{I} : H \to H$ *is a bounded linear operator such that* $\mathcal{I}^2 = -Id$, $\langle z, \mathcal{I}z\rangle = 0, z \in H$. *Let* $\varphi : H \to H$ *be function of the form* $\varphi(z) = \tilde{\varphi}(|z|^2)\mathcal{I}z$, $z \in H$, *where* $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ *is differentiable. Then,* φ *is* $\mathbb{R}$*-differentiable and*

$$(\varphi'(z))(\varphi(z)) = (d_z\varphi)(\varphi(z)) = -|\tilde{\varphi}(|z|^2)|^2 z, \; \forall z \in H. \tag{8.51}$$

Proof. Let $y, z \in H$; then we have $[\varphi'(z)](y) = 2\tilde{\varphi}'(|z|^2)\langle z, y\rangle \mathcal{I} z + \tilde{\varphi}(|z|^2)\mathcal{I} y$. Therefore, given $z \in H$, we deduce

$$\begin{aligned}[\varphi'(z)](\varphi(z)) =& 2\tilde{\varphi}'(|z|^2)\langle z, \tilde{\varphi}(|z|^2)\mathcal{I} z\rangle \mathcal{I} z + \tilde{\varphi}(|z|^2)\mathcal{I}\varphi(z)\\ =& 2\tilde{\varphi}'(|z|^2)\tilde{\varphi}(|z|^2)\langle z, \mathcal{I} z\rangle \mathcal{I} z |\tilde{\varphi}(|z|^2)|^2\mathcal{I}^2 z = -|\tilde{\varphi}(|z|^2)|^2 z.\end{aligned}$$

This completes the proof of (8.51). □

Since we identify $\mathbb{C}$ with $\mathbb{R}^2$ and the operator of multiplication by i with the operator $\mathcal{I} : \mathbb{R}^2 \ni (x, y) \mapsto (-y, x) \in \mathbb{R}^2$, we deduce the following result.

Corollary 8.29. *Assume that a function $g : \mathbb{C} \to \mathbb{C}$ satisfies Assumption 8.27 for a differentiable function $\tilde{g} : \mathbb{R} \to \mathbb{R}$. Then g is $\mathbb{R}$-differentiable and, with $\langle \cdot, \cdot\rangle$ (resp. $|\cdot|$) denoting the scalar product (resp. the euclidian norm), in $\mathbb{C} \cong \mathbb{R}^2$, we have for all $z \in \mathbb{C}$,*

$$m(z) := ((ig)'(z))(ig(z)) = -(\tilde{g}(|z|^2))^2 z, \quad \langle ((ig)'(z))(ig(z)), z\rangle = -|g(z)|^2. \tag{8.52}$$

In particular we get the formulation of $\mathrm{tr}_{\mathrm{K}}(\mathcal{M}(u))$, where $\mathcal{M}$ is defined by (8.39). Let Π be the bilinear map defined in (8.46), $\Lambda : K \to \mathcal{R}$ denotes the natural embedding (which is a gamma-radonifying operator) and $(e_j)_{j\geq 1}$ is a complete orthonormal system of K satisfying (8.36) and consisting of real valued functions. Then it follows from the definition (8.37) of the trace and the Kwapień-Szymański result (8.36) that

$$\mathfrak{p} := \mathrm{tr}_{\mathrm{K}}(\Pi) = \sum_{j\geq 1}(\Lambda e_j)^2 \in H^{s,2}(M, \mathbb{R}) \cap W^{\hat{s},q}(M, \mathbb{R}) \subset \mathcal{R}. \tag{8.53}$$

Let us make another useful observation. Let m be defined in (8.52) and $\mathbb{M}$ be the Nemytski map corresponding to the function $m : \mathbb{C} \to \mathbb{C}$, that is $\mathbb{M}(u) = m \circ u$, $u \in \mathcal{R}$. Then

$$\mathcal{M}(u)(h_1, h_2) = \mathbb{M}(u)h_1h_2, \quad \text{for } u, h_1, h_2 \in \mathcal{R}. \tag{8.54}$$

Furthermore, we have the following:

Lemma 8.30. *Assume that a function $g : \mathbb{C} \to \mathbb{C}$ satisfies Assumption 8.27 for a differentiable function $\tilde{g} : \mathbb{R} \to \mathbb{R}$. Then the maps G and $\mathbb{M}$ transform $\mathcal{R}$ to $\mathcal{L}(\mathcal{R}, \mathcal{R})$ and $\mathcal{R}$ respectively, and, for every $u \in \mathcal{R}$,*

$$\mathrm{tr}_{\mathrm{K}}(\mathcal{M}(u)) = \Pi(\mathfrak{p}, \mathbb{M}(u)), \tag{8.55}$$

(8.56)

$$\mathrm{Re}\,\langle\, \mathrm{tr}_{\mathrm{K}}\mathcal{M}(u), u\rangle_{L^2(M)} = -\int_M |g(u(x))|^2\mathfrak{p}(x)\,dx = -\|G(u)\Lambda\|^2_{R(\mathrm{K},L^2(M))},$$

(8.57)

$$\begin{aligned}\mathrm{Re}\,\langle \nabla\, \mathrm{tr}_{\mathrm{K}}\mathcal{M}(u), \nabla u\rangle =& -\int_M |\tilde{g}(|u(x)|^2)|^2 \mathrm{Re}\,\langle u(x)\nabla\mathfrak{p}(x), \nabla u(x)\rangle\,dx\\ &-4\int_M (\tilde{g}'\tilde{g})(|u(x)|^2)\mathrm{Re}\,\langle u(x)\nabla u(x), \nabla u\rangle\mathfrak{p}(x)\,dx - 2\int_M \tilde{g}(|u(x)|^2)^2\mathfrak{p}(x)|\nabla u(x)|^2\,dx.\end{aligned}$$

Proof. Since both g and m are functions of C^1-class and the point-wise multiplication in $\mathcal{R}$ is a bounded bilinear map, Proposition 8.1 implies that G and $\mathbb{M}$ are well defined maps from $\mathcal{R}$ to $\mathcal{L}(\mathcal{R},\mathcal{R})$ and $\mathcal{R}$. Using (8.39) we deduce that for $u \in \mathcal{R}$,

$$\mathrm{tr}_{\mathrm{K}}\Big[\mathcal{M}(u)\Big] = \sum_{j\geq 1} [((ig)'(ig)) \circ u](\Lambda e_j)^2 \in \mathcal{R}.$$

Then the definition (8.52) of the function m concludes the proof of identity (8.55). Moreover the second identity in (8.52) yields

$$\mathrm{Re}\,\langle\, \mathrm{tr}_{\mathrm{K}}\mathcal{M}(u), u\rangle_{L^2(M)} = \sum_j \int_M \mathrm{Re}\,\langle (ig)'(u(x))(ig(u(x)))((\Lambda e_j)(x))^2, u(x)\rangle dx$$

$$= -\sum_j \int_M ((\Lambda e_j)(x))^2 |g(u(x))|^2\, dx = -\sum_j |\tilde{G}(u)\Lambda e_j|^2_{L^2(M)}.$$

This concludes the proof of (8.56); that of (8.57) is similar. □

Recall that m is defined by (8.52). The above results show that the Stratonovich equation (8.50) can be written in the following Itô form:

$$du(t) = \Big[iAu(t) - if(u(t)) + \frac{1}{2}\mathfrak{p}\, m(u(t))\Big]\, dt - ig(u(t))\, dW(t). \tag{8.58}$$

We now prove the existence and uniqueness of a maximal solution to the problems (8.49) and (8.50) - or (8.58). This is the main result of this section.

Theorem 8.31. *Assume that M is a compact riemannian manifold of dimension $d = 2$. Assume that $f : \mathbb{C} \to \mathbb{R}$ is of real C^1-class satisfying, for some $\beta \geq 2$ and all $y, z \in \mathbb{C}$,*

$$|f(y)| \leq C(1 + |y|^\beta),\ \ |f'(y)| \leq C(1 + |y|^{\beta-1}),$$
$$|f'(y) - f'(z)| \leq C(1 + |y|^{\beta-2} + |z|^{\beta-2})|y - z|. \tag{8.59}$$

Assume that $g : \mathbb{C} \to \mathbb{R}$ is of real C^1-class satisfying, for some $a \geq 1$ and all $y, z \in \mathbb{C}$,

$$|g(y)| \leq C(1 + |y|^a),\ \ |g'(y)| \leq C(1 + |y|^{a-1}),$$
$$|g'(y) - g'(z)| \leq C(1 + |y|^{(a-2)^+} + |z|^{(a-2)^+})|y - z|. \tag{8.60}$$

Assume that $p > \beta \vee (2a)$ and $q > 2$ satisfy the scaling admissible condition (8.45). Assume that $s \in (1 - \frac{1}{p}, 1]$ and let $W = (W(t)\,,\, t \geq 0)$ be an $H^{s,2}(M) \cap W^{s-\frac{1}{p},q}(M)$-valued Wiener process defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual assumptions.
Then for every $u_0 \in L^p(\Omega, \mathcal{F}_0, H^{s,2}(M))$ there exist a local process $u = (u(t)\,,\, t < \tau_\infty)$ whose trajectories are $H^{s,2}(M)$-valued continuous and locally p-integrable with values in $W^{s-\frac{1}{p},q}(M)$, that is the unique local maximal solution to problem (8.49). Moreover, $\mathbb{P}(\{\tau_\infty < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_{H^{s,2}(M)} < \infty\}) = 0$ and

$$\limsup_{t\to\tau_\infty} |u(t)|_{H^{s,2}(M)} = +\infty \ \text{a.s. on } \{\tau_\infty < \infty\}.$$

Suppose furthermore that $p > \beta \vee (2a) \vee \gamma$, that g satisfies Assumption 8.27 and that $m(z) = -\tilde{g}(|z|^2)^2 z$ satisfies the following condition for some $\gamma \geq 2$ and all $y, z \in \mathbb{C}$:

$$|m(y)| \leq C(1 + |y|^\gamma), \;\; |m'(y)| \leq C(1 + |y|^{\gamma-1}),$$
$$|m'(y) - m'(z)| \leq C(1 + |y|^{(\gamma-2)} + |z|^{(\gamma-2)})|y - z|. \tag{8.61}$$

Then the same conclusion as above holds for problem (8.50) in the Stratonovich form.

Remark 8.32. Let $\tilde{g}$ be of class C^2 such that for some constants $C > 0$, $\alpha_0 \geq 0$, and some constants α_i, $i = 1, 2$ one has for all $r \geq 0$:

$$|\tilde{g}(r)| \leq C(1 + r^{\alpha_0}), \quad |\tilde{g}'(r)| \leq C(1 + r^{\alpha_1}), \quad |\tilde{g}''(r)| \leq C(1 + r^{\alpha_2}).$$

Then if $g(z) = \tilde{g}(|z|^2)z$, the function g satisfies condition (8.60) with $a \geq (2\alpha_0 + 1) \vee (2\alpha_1 + 3) \geq 1$ and $a \geq 2\alpha_2 + 5$ if $a > 2$. Furthermore, the function m defined by $m(z) = -\tilde{g}(|z|^2)^2 z$ satisfies (8.61) with $\gamma = (2a - 1) \vee 2$.

8.7. **Existence of a global solution $H^{1,2}$-valued solution to the Stochastic NLS in the Stratonovich form.**

8.7.1. *Preliminaries.* As in section 8.6, we assume below that M is a 2-dimensional, compact riemannian manifold and Δ is the Laplace Beltrami operator on M. In the previous sections we considered the stochastic nonlinear Schrödinger equation (8.49) with the initial data u_0 belonging to the Sobolev space $H^{s,2}(M)$ for some $s \leq 1$. In this section we will consider the problem of global existence for $s = 1$. We at first rewrite the non linear Schrödinger equation (8.49) with a Stratonovich integral and then prove that the $L^2(M)$ norm of the solution is preserved. We finally conclude by means of the Khashmiski Theorem with the energy function playing the rôle of the Lyapunov function.
The following notations already, used in section 8.6 for any $s \in (1 - \frac{1}{p}, 1]$, will be used in the entire section for $s = 1$. Given $\theta \in (0, 1]$ and $r \in [1, +\infty)$, we put $W^{\theta,r}(M) := W^{\theta,r}(M, \mathbb{C})$ where $\mathbb{C}$ is identified to $\mathbb{R}^2$. Let (p, q) be a pair of positive numbers which satisfies the scaling admissible condition (8.45), that is $\frac{2}{p} + \frac{2}{q} = 1$. We set $s = 1$, $\hat{s} = 1 - \frac{1}{p}$, $\mathrm{H} = H^{1,2}(M)$, $\mathrm{E} = W^{\hat{s},q}(M)$, $\|u\|_{\hat{s},q} = \|u\|_{W^{\hat{s},q}(M)}$ and $\mathcal{R} = \mathrm{H} \cap \mathrm{E}$. Since $q > 2$, $\hat{s}q > 2$ and hence we deduce that $W^{\hat{s},q}(M) \subset L^\infty(M)$. We use the notation for the scalar product in $L^2(M) := L^2(M; \mathbb{C})$:

$$\langle u, v\rangle = \int_M \mathrm{Re}\, (u(x)\overline{v(x)})dx, \quad u, v \in L^2(M).$$

We will consider the stochastic NLS equation in Stratonovich form, that is for $u_0 \in H^{1,2}(M)$, the problem

$$idu(t) + \Delta u(t)dt = f(u)dt + g(u) \circ dW(t), \quad u(0) = u_0. \tag{8.62}$$

To prove the global existence of the solution to the NLS equation (8.62), we need to impose conditions on the noise W, on the diffusion coefficient g and on the non-linearity f stonger that those made in the previous section.

Assumption 8.33. *Thus we suppose that $(W(t), t \geq 0)$ is a real $W^{1,2s_0}(M, \mathbb{R}) \cap W^{\hat{s},q}(M, \mathbb{R})$-valued Wiener process, for some $s_0 > 1$.*

Let K be the reproducing kernel Hilbert space of the law of the $H^{1,2s_0}(M,\mathbb{R}) \cap W^{\hat{s},q}(M,\mathbb{R})$-valued random variable $W(1)$. Then the embedding $\Lambda : \mathrm{K} \to H^{1,2s_0}(M,\mathbb{R}) \cap W^{\hat{s},q}(M,\mathbb{R})$ is γ radonifying.

Lemma 8.34. *For $x \in M$ let $\mathfrak{p}(x) = \mathrm{tr}_K(\Pi)(x) = \sum_j |\Lambda e_j(x)|^2$ and $\mathfrak{q}(x) = \sum_{j\geq 1} |\nabla \Lambda e_j(x)|^2$. Then $\mathfrak{p} \in L^\infty(M)$ and $\mathfrak{q} \in L^1(M)$. Furthermore, $\sum_{j\geq 1} \|\nabla \Lambda e_j\|^2_{L^{2s_0}} < \infty$.*

Proof. By the Kwapień-Szymański Theorem [31], we can assume that the ONB $\{e_j\}_{j=1}^\infty$ is chosen in such a way that $\sum_{j\geq 1} \|\Lambda e_j\|^2_{H^{1,2s_0}(M)\cap W^{\hat{s},q}(M)} < \infty$. Since $W^{\hat{s},q}(M) \subset L^\infty(M)$ we deduce that $\mathfrak{p}$ is bounded. Furthermore, $\sum_{j\geq 1} \|\nabla \Lambda e_j\|^2_{L^1(M)} \leq \sum_{j\geq 1} \|\nabla \Lambda e_j\|^2_{L^{s_0}(M)} < \infty$ and therefore, the series $\sum_{j\geq 1} |\nabla \Lambda e_j|^2$ is absolutely convergent in $L^1(M)$ as claimed; this concludes the proof. ◻

In this section, we suppose that g satsfies the following stronger version of Assumption 8.27.

Assumption 8.35. *There exists a bounded function $\tilde{g} : [0,+\infty) \to \mathbb{R}$ of class C^1 such that*

$$g(z) = \tilde{g}(|z|^2)z, \; z \in \mathbb{C}, \tag{8.63}$$

Furthermore, we assume that the function g satisfies the conditions (8.60) *with $a = 1$ and the function $m : \mathbb{C} \to \mathbb{C}$ defined by $m(z) = -\tilde{g}(|z|^2)^2 z$ satisfies condition* (8.61) *with $\gamma \geq 2$.*

An example of function $\tilde{g}$ such that the function g defined by (8.63) satisfies Assumption 8.35 is a bounded function of class C^2 such that $\sup_{r>0}(1 + r)|\tilde{g}'(r)| < \infty$ and $\sup_{r>0} r^{\frac{3}{2}}|\tilde{g}''(r)| < \infty$, for instance, $\tilde{g}(r) = \frac{\ln(1+r)}{C+\ln(1+r)}$ for $r > 0$ and $C > 0$. Indeed, the conditions in Remark 8.32 are satisfied with $\alpha_0 = 0$, $\alpha_1 = -1$ and $\alpha_2 = -\frac{3}{2}$, which yields (8.60) for g with $a = 1$ while m satisfies (8.61) with $\gamma = 2$.

We study the global existence for two types of problem (8.62) depending on the non-linear term f, which is either defocusing or focusing. Precise assumptions will be described below, but let us mention that a typical example of the former is when $f(u) = |u|^2u$ while a typical example of the latter is when $f(u) = -|u|u$.

Assumption 8.36. *We assume that $f : \mathbb{C} \to \mathbb{R}$ is of the form*

$$f(z) = \tilde{f}(|z|^2)z, \; z \in \mathbb{C}, \tag{8.64}$$

where the function $\tilde{f} : [0,+\infty) \to \mathbb{R}$ satisfies one of the following two cases.

Case 1: defocusing nonlinearity. *The function $\tilde{f}$ satisfies either (a) or (b):*
(a) *There exist a natural number N and real number a_k, $k = 0, \cdots, N$, with $a_N > 0$, such that $\tilde{f}(r) = \sum_{k=0}^N a_k r^k$ for every $r \in \mathbb{R}$.*
(b) *There exist $C > 0$ and $\sigma \in [\frac{1}{2}, \infty)$ such that $\tilde{f}(r) = Cr^\sigma$ for every $r \in \mathbb{R}$.*

Case 2: focusing nonlinearity. *There exist $C > 0$ and $\sigma \in [\frac{1}{2}, 1)$ such that for every $r \in \mathbb{R}$, $\tilde{f}(r) = -Cr^\sigma$.*

8.7.2. *Preservation of the L^2-norm.* We at first prove that the $L^2(M)$-norm of this solution is almost surely constant in time. This extends classical results for the deterministic NLS equation (see [16] for the case of compact manifolds) as well as [19] de Bouard Debussche for the flat stochastic NLS equation.

Lemma 8.37. *Assume that f and g satisfy the Assumptions 8.36 and 8.35 respectively, p and q satisfy the scaling admissibility condition $\frac{2}{p}+\frac{2}{q}=1$. Let $(W(t), t \geq 0)$ be an* $\mathrm{H} \cap \mathrm{E}$*-valued Wiener process and $u_0 \in H$. Then $|u(t)|_{L^2(M)} = |u_0|_{L^2(M)}$, for all $t \in [0, \tau_\infty)$, $\mathbb{P}$-almost surely.*

Proof. Let $(\tilde{\tau}_k)_k$ denote the approximating sequence of the stopping time τ_∞ defined by

$$\tilde{\tau}_k := \inf\{t \in [0, \tau_\infty) : |u(t)|_{H^{1,2}} \geq k\}. \tag{8.65}$$

Suppose that we have proved that for each $t \geq 0$ and $k \in \mathbb{N}$, $|u(t \wedge \tilde{\tau}_k)|_{L^2(M)} = |u_0|_{L^2(M)}$ $\mathbb{P}$-almost surely. Then it follows that there exists a set $\hat{\Omega} \subset \Omega$ of full $\mathbb{P}$-measure such that for each $\omega \in \hat{\Omega}$, $|u(t, \omega)|_{L^2(M)} = |u_0|_{L^2(M)}$ for all $t \in \mathbb{Q} \cap [0, \tau(\omega))$. Thus, since for all $\omega \in \hat{\Omega}$ the map $[0, \tau(\omega)) \ni t \mapsto u(t, \omega) \in L^2(M)$ is continuous, the result will follow.
To prove the conservation of the $L^2(M)$-norm, let us consider the functional

$$\Phi : L^2(M) \ni u \mapsto \frac{1}{2}|u|^2_{L^2(M)} = \frac{1}{2}\int_M u(x)\overline{u(x)}\, dx \in \mathbb{R},$$

where dx denotes the integration with respect to the riemannian volume measure on M. The function Φ is of real-C^∞ class and for all $u, v, v_1, v_2 \in L^2(M)$, we have

$$\begin{aligned}
\Phi'(u)(v) &= d_u\Phi(v) = \mathrm{Re}\,\langle u, v\rangle_{L^2} = \int_M \mathrm{Re}\,(u(x)\overline{v(x)})\, dx,\\
\Phi''(u)(v_1, v_2) &= d_u^2\Phi(v_1, v_2) = \mathrm{Re}\,\langle v_1, v_2\rangle_{L^2} = \int_M \mathrm{Re}\,(v_1(x)\overline{v_2(x)})\, dx.
\end{aligned}$$

Let us now assume, for purely pedagogical reasons, that u is a strong solution. Applying the Itô formula we obtain for each $t \in \mathbb{R}_+$ and every $k \in \mathbb{N}$, $\mathbb{P}$-almost surely,

$$\begin{aligned}
\Phi(u(t\wedge\tilde{\tau}_k))-\Phi(u_0) &= -\int_0^t 1_{[0,\tilde{\tau}_k)}(s)\langle\Phi'(u(s)), iG(u(s))\rangle_{L^2}\,dW(s)\\
&\quad+\int_0^t 1_{[0,\tilde{\tau}_k)}(s)\Big\langle\Phi'(u(s)), i[\Delta u(s)-F(u(s))]+\frac{1}{2}\mathrm{tr}_K(\Pi)\mathbb{M}(u(s))\Big\rangle_{L^2}ds\\
&\quad+\frac{1}{2}\int_0^t 1_{[0,\tilde{\tau}_k)}(s)\mathrm{tr}_K\Big[\Phi''(u(s))(iG(u(s)), iG(u(s)))\Big]ds\\
&=\int_0^t 1_{[0,\tilde{\tau}_k)}(s)\mathrm{Re}\,\langle u(s), i\Delta u(s)\rangle_{L^2}\,ds-\int_0^t 1_{[0,\tilde{\tau}_k)}(s)\mathrm{Re}\,\langle u(s), iF(u(s)\rangle_{L^2}\,ds\\
&\quad+\frac{1}{2}\mathfrak{p}\int_0^t 1_{[0,\tilde{\tau}_k)}(s)\mathrm{Re}\,\langle u(s),\mathbb{M}(u(s))\rangle_{L^2}\,ds\\
&\quad-\int_0^t 1_{[0,\tilde{\tau}_k)}(s)\mathrm{Re}\,\langle u(s), iG(u(s))\rangle_{L^2}\,dW(s)+\frac{1}{2}\int_0^t 1_{[0,\tilde{\tau}_k)}(s)\sum_{j\ge1}|G(u(s))\Lambda e_j|^2_{L^2}\,ds.
\end{aligned}$$

Next we make the following three observations.

(1) Since Δ is self-adjoint in $L^2(M)$, we have $\mathrm{Re}\,\langle u(s), i\Delta u(s)\rangle_{L^2(M)}=0$.

(2) If H is the Nemytski map associated with h of the form $h(z)=\tilde{h}(|z|^2)z$, where $\tilde{h}:\mathbb{R}\to\mathbb{R}$, then

$$\mathrm{Re}\,\langle u(s), iH(u(s)\rangle_{L^2}=\int_M\tilde{h}(|u(s,x)|^2)\mathrm{Re}\,[u(s,x)\overline{iu(s,x)}]\,dx=0.$$

(3) Lemma 8.30 implies that $\mathfrak{p}\mathrm{Re}\,\langle u(s),\mathbb{M}(u(s))\rangle_{L^2}=-\sum_{j\ge1}|G(u(s))\Lambda e_j|^2_{L^2}$.

Therefore, we infer that that for each $t\ge0$ and every $k\in\mathbb{N}$, $\mathbb{P}$-almost surely, $\Phi(u(t\wedge\tau_k))-\Phi(u_0)=0$, that is $|u(t\wedge\tilde{\tau}_k)|_{L^2(M)}=|u_0|_{L^2(M)}$ $\mathbb{P}$-almost surely and the result follows.

A full proof can be made by replacing u by its Yosida approximation as it has been done for instance in [10]; □

8.7.3. *The Lyapounov function.* As in the deterministic case, we will use some Lyapunov function. Let $\tilde{F}$ denote the antiderivative of $\tilde{f}$ such that $\tilde{F}(0)=0$. In this section, we will consider two cases as in Assumption 8.36.

Case 1(a). We assume that $\tilde{f}$ is a polynomial of degree N with a positive leading coefficient. Hence $\tilde{F}(r)=a_{N+1}r^{N+1}+Q(r)$, where Q is a polynomial function of degree at most N such that $Q(0)=0$ and $a_{N+1}>0$. We have the following result.

Lemma 8.38. *Let $\tilde{F}$ and Q be polynomial functions as above. Then there exists a constant $C>0$, and for every $\varepsilon>0$, there exists a constant $C(\varepsilon)>0$ such that for all $u\in L^{2N+2}(M)(\supset\mathcal{R})$, we have*

$$\int_M|u(x)|^{2N+2}dx\le C\int_M|\tilde{F}(|u(x)|^2)|dx+C\int_M|u(x)|^2dx, \tag{8.66}$$

$$\Big|\int_M Q(|u(x)|^2)\,dx\Big|\le\varepsilon\int_M|u(x)|^{2N+2}dx+C(\varepsilon)\int_M|u(x)|^2dx, \tag{8.67}$$

while for $u \in H^{1,2}(M)$ we have

$$\int_M |\tilde{f}(|u(x)|^2)|u(x)|^2\,dx \le C\int_M |\tilde{F}(|u(x)|^2)|\,dx + C\int_M |u(x)|^2\,dx. \tag{8.68}$$

Case 1(b). We assume that $\tilde{f}(r) = Cr^\sigma$ for $C > 0$ and $\sigma \ge \frac{1}{2}$. Then $\tilde{F} \ge 0$ and thus

$$\int_M \tilde{F}(|u(x)|^2)dx \ge 0, \text{ for any } u \in \mathcal{R}.$$

Furthermore, since $r\tilde{f}(r) = C\tilde{F}(r)$, there exists some positive constant C such that

$$\int_M \tilde{f}(|u(x)|^2)|u(x)|^2\,dx \le C\int_M |\tilde{F}(|u(x)|^2)|\,dx, \quad \forall u \in H^{1,2}(M). \tag{8.69}$$

Case 2. We assume that $\tilde{f}(r) = -Cr^\sigma$ for $C > 0$ and $\sigma \in [\frac{1}{2}, 1)$. Then $\tilde{F}(r) = -\frac{C}{\sigma+1}r^{\sigma+1}$. The following lemma will be used to deal with this case.

Lemma 8.39. *Assume that $\alpha \in (1,3)$. Then for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that for any $u \in H^{1,2}(M)$ (and hence for any $u \in \mathcal{R}$),*

$$\int_M |u(x)|^{\alpha+1}dx \le \varepsilon|\nabla u|^2_{L^2} + C(\varepsilon)|u|^{\frac{4}{3-\alpha}}_{L^2}. \tag{8.70}$$

Furthermore, for $\sigma \in [\frac{1}{2}, 1)$ there exists $\tilde{C} > 0$ such that for every $u \in H^{1,2}(M)$,

$$\int_M |\tilde{f}(|u(x)|^2)||u(x)|^2\,dx \le \frac{1}{2}|\nabla u|^2_{L^2} + \frac{1}{2}\int_M \tilde{F}(|u(x)|^2)dx + \tilde{C}|u|^{\frac{2}{1-\sigma}}_{L^2}. \tag{8.71}$$

Proof. The Gagliardo-Nirenberg and Young inequalities imply that for $u \in H^{1,2}(M)$ and $\varepsilon > 0$,

$$\int_M |u(x)|^{\alpha+1}dx \le C\,|\nabla u|^{\alpha-1}_{L^2}\,|u|^2_{L^2} \le \varepsilon|\nabla u|^2_{L^2} + C\varepsilon^{-\frac{2(\alpha-1)}{3-\alpha}}\,|u|^{\frac{4}{3-\alpha}}_{L^2}.$$

This proves inequality (8.70). Finally, $|\tilde{f}(r)|r = cr^{\sigma+1}$, $r \ge 0$ and since $\sigma < 1$ we have

$$\int_M |\tilde{f}(|u(x)|^2)||u(x)|^2\,dx - \frac{1}{2}\int_M \tilde{F}(|u(x)|^2)dx = C\Big(1 + \frac{1}{2(\sigma+1)}\Big)\int_M |u(x)|^{2+2\sigma}\,dx.$$

Hence using (8.70) with $\varepsilon = \frac{1+\sigma}{2C(2\sigma+3)}$ we conclude the proof. □

8.8. **Existence of a global solution.** We can now state the main result of this section, proving that the non linear stochastic Schrödinger equation (8.62) has a unique global solution.

Theorem 8.40. *Assume that the function $\tilde{f}$ satisfy Assumption 8.36, that g satisties Assumption 8.35 and that the Wiener process $(W(t), t \ge 0)$ satisfies Assumption 8.33. Let Let $\beta = 2N+1 \ge 2$ in the* ***defocusing*** *case 1(a) and with $\beta = 2\sigma+1 \ge 2$ in the focusing case 2 or the defocusing case 1(b). , $p > \beta \vee \gamma$ where γ is defined in Assumption 8.35, and let q be such that (p,q) satisfy the scaling admissible condition $\frac{2}{p} + \frac{2}{q} = 1$. Suppose furthermore that $u_0 \in H^{1,2}(M)$. Then the stochastic NLS equation (8.62) has a unique global solution whose trajectories belong a.s. to $C([0,\infty), H^{1,2}(M))$.*

Let us assume that $\tilde{f}$ satisfies Assumption 8.36, hence either the conditions of Case **1(a)**, **1(b)** or **2** above. Let us define a map

$$\Psi : \mathcal{R} \ni u \mapsto \frac{1}{2}|\nabla u|_{L^2}^2 + \frac{1}{2}\int_M \tilde{F}(|u(x)|^2)dx \in \mathbb{R}. \tag{8.72}$$

Using Lemmas 8.38 or 8.39, it is easy to see that there exists a constant $C > 0$ such that $\Psi(u) + C|u|_{L^2}^2 \geq 0$ for all $u \in \mathcal{R}$. This proves the following

Corollary 2. *There exists a constant $c \geq 0$ such that*

$$|u|_{H^{1,2}}^2 \leq 2\Psi(u) + c|u|_{L^2}^2, \quad u \in \mathcal{R}.$$

We will need the following result about the regularity of Ψ and some of its properties.

Lemma 8.41. *The function Ψ defined by (8.72) is of real C^2-class with the second derivative bounded on balls; for all $u, v_1, v_2 \in \mathcal{R}$, we have*

$$\begin{aligned}
\Psi'(u)(v) &= \mathrm{Re} \int_M \nabla u(x)\overline{\nabla v(x)}\, dx + \int_M \tilde{f}(|u(x)|^2)\, \mathrm{Re}\, [u(x)\overline{v(x)}]\, dx, \\
\Psi''(u)(v,v) &= \int_M |\nabla v(x)|^2\, dx + \int_M \tilde{f}(|u(x)|^2)|v(x)|^2 dx \\
&\quad +2\int_M \tilde{f}'(|u(x)|^2)\Big(\mathrm{Re}\ [u(x)\overline{v(x)}]\Big)^2\, dx.
\end{aligned}$$

Moreover,

$$\langle \Psi'(u), i[\Delta u - F(u)]\rangle = 0, \quad u \in H^{2,2}(M), \tag{8.73}$$

$$\langle \Psi'(u), iG(u)\rangle = \int_M \mathrm{Re}\ (\nabla u(x)\overline{\nabla ig(u(x))})dx, \quad u \in \mathcal{R}, \tag{8.74}$$

$$\begin{aligned}
\mathrm{tr}_K \Psi''(u)(iG(u), iG(u)) &\leq 2\int_M |g'(u(x))\nabla u(x)|^2 \mathfrak{p}(x)\, dx \\
&+ 2\int_M |g(u(x))|^2 \mathfrak{q}(x)\, dx \\
&+ \int_M \tilde{f}(|u(x)|^2)|g(u(x))|^2 \mathfrak{p}(x)\, dx \quad u \in \mathcal{R},
\end{aligned} \tag{8.75}$$

where $\mathfrak{p}(x)$ and $\mathfrak{q}(x)$ are defined in Lemma 8.34.

The following lemma gives an explicit expression of $\Psi(u(t))$, where $(u(t), t \in [0, \tau_\infty))$ denotes the local maximal solution to (8.62). Note that, unlike in the deterministic case, the Itô-Stratonovich correction term yields that $\mathbb{E}(\Psi(u(t)))$ is not time invariant.

Lemma 8.42. *Assume that $(W(t), t \geq 0)$ is an $\mathcal{R}$-valued Wiener process. Then in the framework above, for every $t \geq 0$ and every $k \in \mathbb{N}^*$, we have*

$$\Psi(u(t\wedge\tilde{\tau}_k)) = \Psi(u_0) - \int_0^{t\wedge\tilde{\tau}_k}\int_M \mathrm{Re}\ (\nabla u(s,x)\overline{\nabla ig(u(s,x))})\, dx\, dW(s) + \mathcal{T}(t\wedge\tilde{\tau}_k), \tag{8.76}$$

where

$$\mathcal{T}(t\wedge\tilde\tau_k) \le \int_0^{t\wedge\tilde\tau_k}\int_M |g'(u(s,x))|^2\,|\nabla u(s,x)|^2\,\mathfrak{p}(x)\,dx\,ds + \int_0^{t\wedge\tilde\tau_k}\int_M |g(u(s,x))|^2\,\mathfrak{q}(x)\,dx\,ds$$

(8.77)

$$+\frac{1}{2}\int_0^{t\wedge\tilde\tau_k}\int_M \tilde f(|u(s,x)|^2)|g(u(s,x))|^2\mathfrak{p}(x)\,dx\,ds$$

8.8.1. *Existence of a global solution.* We can now state the main result of this section, proving that the non linear stochastic Schrödinger equation (8.62) has a unique global solution.

Theorem 2. Assume that the function $\tilde f$ satisfy Assumption 8.36, that g satisfies Assumption 8.35 and that the Wiener process $(W(t), t\ge 0)$ satisfies Assumption 8.33. Let $\beta = 2N+1\ge 2$ in the defocusing case 1(a) and with $\beta = 2\sigma+1\ge 2$ in the focusing case 2 or the defocusing case 1(b). , $p>\beta\vee\gamma$ where γ is defined in Assumption 8.35, and let q be such that (p,q) satisfy the scaling admissible condition $\frac{2}{p}+\frac{2}{q}=1$. Suppose furthermore that $u_0\in H^{1,2}(M)$. Then the stochastic NLS equation (8.62) has a unique global solution whose trajectories belong a.s. to $C([0,\infty), H^{1,2}(M))$.

Proof. Let $u=(u(t)\,,\, t<\tau_\infty)$ belonging to $\mathbb{M}^p_{\text{loc}}(Y_{[0,\tau_\infty)})$, be the unique local maximal solution to the problem (8.62). Note that $\limsup_{t\to\tau_\infty}|u(t)|_{H^{1,2}} = +\infty$ a.s. on $\{\tau_\infty<\infty\}$; for an integer $k\ge 1$ recall that $\tilde\tau_k = \inf\{t\ge 0 : |u(t)|_{H^{1,2}}\ge k\}$. Using the Khashminskii test for non-explosions (see [29, Theorem III.4.1] for the finite-dimensional case) and arguing as in [10, page 7] it is sufficient to show that each $t>0$, there exists a constant $C_t>0$ such that

$$\mathbb{E}\Big(|u(t\wedge\tilde\tau_k)|^2_{H^{1,2}}\Big)\le C_t,\quad \text{for every } k\in\mathbb{N}^*. \tag{8.78}$$

In view of Corollary 2 and Lemma 8.37 it is sufficient to find, for each $t>0$, a constant $C_t>0$ such that

$$\mathbb{E}\Big(\Psi(u(t\wedge\tilde\tau_k))\Big)\le C_t,\quad \text{for every } k\in\mathbb{N}^*. \tag{8.79}$$

Since $W^{1,2s_0}(M,\mathbb{R})\cap W^{\hat s,q}(M,\mathbb{R})\subset\mathcal{R}$, the assumptions of Lemma 8.42 are satisfied. Hence, for each $t\in\mathbb{R}_+$ and every $k\in\mathbb{N}$, $\mathbb{P}$-almost surely,

$$\begin{aligned}\mathbb{E}\Psi(t\wedge\tilde\tau_k)\le{}&\mathbb{E}\Psi(u_0)+C|\mathfrak{p}|_\infty\mathbb{E}\int_0^{t\wedge\tilde\tau_k}\int_M|\nabla u(s,x)|^2\,dx\,ds\\&+C\sum_{j\ge1}\mathbb{E}\int_0^{t\wedge\tilde\tau_k}\int_M|u(s,x)|^2|\nabla\Lambda e_j(x)|^2\,dx\,ds\\&+C|\mathfrak{p}|_\infty\,\mathbb{E}\int_0^{t\wedge\tilde\tau_k}\int_M\tilde f(|u(s,x)|^2)|u(s,x)|^2\,dx\,ds.\end{aligned} \tag{8.80}$$

Let s_0^* denote the conjugate exponent to s_0. The Gagliardo-Nirenberg inequality proves that $H^{1,2}(M)\subset L^{2s_0^*}(M)$; Hölder's inequality, Lemma 8.34 and Corollary 2 imply that for $u \in H^{1,2}(M)$,

$$\sum_{j\geq 1}\int_M |u(x)|^2|\nabla\Lambda e_j(x)|^2\,dx \leq \|u\|^2_{L^{2s_0^*}}\sum_j \|\nabla\Lambda e_j\|^2_{L^{2s_0}} \leq C[\Psi(u)+|u|_{L^2(M)}].$$

Next, the inequalities (8.68), (8.69) and (8.71) imply the existence of positive constants C and δ such that for all $u \in H^{1,2}(M)$,

$$\int_M \tilde{f}(|u(x)|^2)|u(x)|^2\,dx \leq C[\Psi(u)+|u|^{\delta}_{L^2(M)}].$$

Therefore, the conservation of energy proved in Lemma 8.37 and the above estimates imply the existence of an increasing function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ and a constant $C_t>0$ such that

(8.81)

$$\begin{aligned}\mathbb{E}\Psi(t\wedge\tilde{\tau}_k) \leq & \mathbb{E}\Psi(u_0)+C\mathbb{E}\int_0^{t\wedge\tilde{\tau}_k}\Psi(u(s))\,ds + C\mathbb{E}\int_0^{t\wedge\tilde{\tau}_k}\phi(|u(s\wedge\tilde{\tau}_k)|_{L^2(M)})\,ds\\ \leq & \mathbb{E}\Psi(u_0)+C\mathbb{E}\int_0^{t}\Psi(u(s\wedge\tilde{\tau}_k))\,ds + C\phi(|u_0|_{L^2(M)}).\end{aligned}$$

The Gronwall Lemma yields that for some constant $C>0$ the upper estimate

$$\mathbb{E}\Psi(t\wedge\tilde{\tau}_k) \leq [\mathbb{E}\Psi(u_0)+Ct\phi(|u_0|_{L^2(M)})]e^{Ct},\quad t\geq 0, \tag{8.82}$$

holds for every integer $k \geq 1$. This concludes the proof of inequality (8.79) and hence that of the Theorem. □

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