On direct approach to diffusion filtering SPDEs and on stability problems

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(2nd part of this work is joint with Marina Kleptsyna, Le Mans)

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Abstract

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Ia: Backward SPDE for diffusion filtering model

Ib: Backward SPDE for solution of SDE

Ic: Backward SPDE for diffusion filtering model (continued)

IIa: Stability of filtering measure for the “ergodic” signal under non-specified initial data

IIb: Technicalities: symmetrisation & continuity in Y

This file contains parts Ia, Ib, and Ic.

Ia: diffusion filtering SPDE.

Apparently, the theory of SPDEs has been boosted in the 70s and 80s after the works by E. Pardoux and N. V. Krylov & B. L. Rozovsky. Hence, in my lectures I return to the nearly beginning and in the 1st part of them (Ia–Ib–Ic) will show how to derive a filtering SPDE “by hands”. All possible simplifications will be made like 1D, smoothness, diffusion coefficients are equal to one.

The second part of the lectures (IIa – IIb) will tackle the problem of the long run of the filtering algorithm with *wrong initial distributions* in the case of the *ergodic signal*.

Diffusion filtering model

Consider an SDE system with smooth bounded coefficients

$$\begin{aligned} dX_t &= f(X_t)dt + dW_t^1, \quad X_0 = x, \\ dY_t &= h(X_t)dt + dW_t^2, \quad Y_0 = y, \end{aligned} \tag{1}$$

where W^1, W^2 are independent Wiener processes. Here X is a *signal* and Y an observation process. The first goal and the first part of this presentation concerns to the equation on the process

$$m_t := E[g(X_t)|\mathcal{F}_t^Y], \, 0 \le t \le T,$$

for some given function g and $T > 0$; this is the best in square mean estimation of $g(X_t)$ given the observations $Y_s: \ 0 \le s \le t$. For simplicity, both SDEs in this talk are in R^1, although any finite dimension may be allowed.

Of Y and Girsanov's change of measure

In the sequel stochastic integrals over Y_t will appear; in this respect recall that under our assumptions on any bounded interval the distribution of Y is equivalent to the distribution of a Wiener process. In other words, after Girsanov's change of measure Y itself may be regarded as a Wiener process.

In particular, stochastic integration may be considered with respect to Y as if it were a Wiener process.

"Backward" stochastic integral below will simply mean a usual integration in the reversed time.

Linear SPDE: we are to show the following

Theorem

Let $g(x) \equiv x$. The process $m_T = E[g(X_T)|\mathcal{F}_T^Y]$ admits the following representation:

$$m_T = \frac{v^g(0,x)}{v^1(0,x)},$$

where $v^g(t,x)$ is a solution of the SPDE

$$\begin{aligned} -d_t v^g(t,x) &= \left[\tfrac{1}{2} v^g_{xx}(t,x) + f(x) v^g_x(t,x)\right] dt \\ &+ h(x) v^g_x(t,x) \star dY_t, \quad 0 \le t \le T, \end{aligned} \tag{2}$$

with initial data

$$v^g(T,x) = g(x), \quad x \in R^1.$$

Backward SPDE: comments

The equation

$$-dv^g(t,x) = \left[\frac{1}{2} v^g_{xx}(t,x) + f(x) v^g_x(t,x)\right] dt$$
$$+h(x) v^g_x(t,x) \star dY_t, \quad 0 \le t \le T,$$

with initial data

$$v^g(T,x) = g(x) \equiv x, \quad x \in R^1,$$

is understood in the following sense: here "$\star$" in $h(x)v^g_x(t,x) \star dY_t$ signifies that we integrate "backward" in time. Note that $v^g(t,x), v^g_x(t,x) \in \mathcal{F}^Y_{t,T}$ and hence v^g_x *may* be integrated in this way by changing direction of time, say, over a *new Wiener process* $V_t := Y_{T-t}$ under a new measure. *In reversed time this is a usual Itô's integral.*

Remark: of Itô's integral in reversed time

Remark. For $0 \le t \le T$, the SPDE (10) may be written in the integral form as

$$v^g(t,x) - v^g(T,x) = \int_t^T \left[\frac{1}{2} v^g_{xx}(s,x) + f(x) v^g_x(s,x)\right] ds$$

$$+ \int_t^T h(x) v^g_x(s,x) \star dY_s.$$

If we let $\hat{Y}_s = Y_{T-s} - Y_T$, $\hat{v}^g(s,x) = v^g(T-s,x)$, then the stochastic integral above may be rewritten as

$$\int_t^T h(x) v^g_x(s,x) \star dY_s = \int_0^{T-t} h(x) \hat{v}^g_x(s,x) d\hat{Y}_s,$$

i.e., as a "usual" Itô's integral.

Independent case $h = 0$: backward PDE

Remark. Note that (10) is a stochastic version of a "backward" parabolic PDE. If $h \equiv 0$ then Y and X are independent and $v^1(0, x) \equiv 1$, so, we have,

$$m_{0,x;T} = m_T = E[g(X_T)|\mathcal{F}_T^Y] = Eg(X_T) = E_{0,x}g(X_T).$$

In this case it is well-known that for $0 \le s \le T$, the function[2] $m_{s,x;T} := E_{s,x}g(X_T)$ satifies the backward Kolomogorov equation,

$$-dv^g(t,x) = \left[\frac{1}{2}v^g_{xx}(t,x) + f(x)v^g_x(t,x)\right] dt,\ 0 \le t \le T,$$

with "initial" data

$$v^g(T,x) = g(x),\ x \in R^1.$$

[2] Here $(0, x)$ in $E_{s,x}$ means initial data x at initial time s for the process.

Ib: Auxiliary: SPDE on $X_t^{s,x}$

Observations Y may be regarded as some incomplete information. In the case of complete information, however, we also have an SPDE established by Krylov and Rozovsky in 1980-1982. Apparently, at least, informally, this may be regarded as a limiting case with $h(x) = X$ and diffusion $\epsilon \to 0$ in a more involved problem with interdependence of W^1 and W^2, which we do not consider here.

On SDE for the signal X

We start with the equation on the signal itself,

$$dX_t = f(X_t)dt + dW_t^1, \; t \geq 0; \quad X_0 = x.$$

For a while we drop the index " 1" in the Wiener process (W^1) and study the solution of this SDE. In fact, it is useful to extend the problem and to consider the following version:

$$\boxed{dX_t = f(X_t)dt + dW_t, \; t \geq s; \qquad X_s = x} \tag{3}$$

Strong solution of this equation will be denoted by $\boxed{X_t^{s,x}}$. Under certain smoothness assumptions it is continuous in s and x (and it is always continuous in t), which may be established via Kolmogorov's continuity theorem.

Continuity via Kolmogorov's theorem

Why this is important? Because we want to write down for any $0 \le s \le t$, the "evolution property" (in fact, Markov property) as

$$X_t^{0,x} = X_t^{s,X_s^{0,x}}. \qquad (4)$$

Continuity in all arguments allows us such substitution of $X_s^{0,x}$ into the initial value of $X_t^{s,\cdot}$. The equation (4) itself follows easily from the uniqueness of solution of the related SDEs and, indeed, from Markov property. The latter is a standard feature of any SDE with a unique solution in strong or weak sense.

How to verify continuity: (on the board?)

NB: about strong solution of (3)

Remark. As a little digression, we note that the equation (3) has a pathwise unique strong solution for any bounded Borel drift f; also for any f satisfying a linear growth bound condition. For 1D case the reference is [A. K. Zvonkin, 1974]; for $D > 1$ [A. Yu. Veretennikov, 1980]. In 2006 this result was extended to locally integrable drift [N. V. Krylov & M. Röckner, 2005], and more extensions appeared more recently including SDEs in Hilbert spaces. Since we assume f smooth, it suffices here to use Itô's original result.

SPDE on $X_t^{s,x}$ –1

Recall the following Krylov and Rozovsky's result concerning multidimensional SDEs. Consider the family of processes $(Z_t^{s,z},\ t \geq s \geq 0,\ z \in R^d)$ which satisfy the *multidimensional* SDEs:

$$dZ_t^{s,z} = b(Z_t^{s,z})dt + \sigma(Z_t^{s,z})dW_t, \quad t \geq s, \quad Z_s^{s,z} = z, \quad (5)$$

where b is a bounded smooth d-dimensional vector, σ is a matrix $d \times d_1$, w_t is a d_1-dimensional Wiener process. We'll use the following different notations for the same value:

$$Z_t^{s,z} \equiv Z(s,t,z),$$

and for $t = T$ also

$$Z_T^{s,z} = u(s,z).$$

SPDE on $X_t^{s,x}$ –2

Theorem (Krylov, Rozovsky 1982)

The function $u(s,z)$ satisfies an SPDE

$$-du(t,z) = [(1/2)(\sigma\sigma^*)_{ij}(z)u_{z_iz_j}(t,z) + b^i(z)u_{z_i}(t,z)]dt + \sigma(z)_{ij}u_{z_i}(t,z) \star dw_t^j \qquad (6)$$

with "initial data" (actually, terminal one)

$$u(T,z) \equiv z. \qquad (7)$$

The (1–dimensional) equation (6) holds true for each component of the vector $u(t,z)$.

Below we will show a "direct proof" of this result, since it is important for the sequel. "Clearly", (7) is straightforward.

SPDE on $X_t^{s,x}$ –3

Due to [Freidlin 1962] or [Kunita 1979][3] the random field $X_t^{s,x}$ is continuous in (s,t,x) for $s \le t$ and $x \in R^d$ and, moreover, it admits continuous partial derivatives $\partial_x X_t^{s,x} =: X_x(x,s,t)$ and $\partial_{xx}^2 X_t^{s,x} =: X_{xx}(x,s,t)$. What is stated in the Theorem,

$$-du(t,z) = [(1/2)(\sigma\sigma^*)_{ij}(z)u_{z_i z_j}(t,z) + b^i(z)u_{z_i}(t,z)]dt \\ +\sigma(z)_{ij}u_{z_i}(t,z) \star dw_t^j,$$

may be written in the integral form with $a = \sigma\sigma^*/2$,

$$X(x,t,T) - X(x,T,T) = \int_t^T X_x(x,s,T)\sigma(x) \star dW_s \\ + \int_t^T [X_x(x,s,T)b(x) + Tr\, X_{xx}(x,s,T)a(x)]\, ds. \quad (8)$$

[3] with relaxed assumptions

SPDE on $X_t^{s,x}$ –3

In the sequel, assume $d = 1$ for simplicity of presentation; but we will use it in the $2D$ case later. To show (8) we split the interval $[0, t]$ by small partitions $t = t_0 < t_1 < \cdots < t_{n+1} = T$, and write down,

$$X(x,t,T) - X(x,T,T) = \sum_{i=0}^{n} \left(X(x,t_i,T) - X(x,t_{i+1},T)\right),$$

and consider each term substituting

$$X(x,t_i,T) = X(X(x,t_i,t_{i+1}),t_{i+1},T),$$

and using Hadamard's form of Newton–Leibnitz' formula,

$$F(\Delta) = F(0) + \int_0^1 F'(\alpha\Delta)\Delta d\alpha$$

$$= F(0) + \Delta \int_0^1 \left(F'(0) + \alpha\Delta \int_0^1 F''(\alpha\beta))d\beta\right) d\alpha.$$

SPDE on $X_t^{s,x}$ –3

Namely, we write,

$$X(X(x,t_i,t_{i+1}),t_{i+1},T)-X(x,t_{i+1},T)=X_x(x,t_{i+1},T)z_i$$
$$+\int_0^1\int_0^1 \alpha X_{xx}(x+\alpha\beta z_i,t_{i+1},T)z_i^2\,d\alpha\,d\beta,$$

where

$$z_i = X(x,t_i,t_{i+1}) - x.$$

Let also

$$\tilde{z}_i = \sigma(x)(W_{t_{i+1}} - W_{t_i}) + b(x)(t_{i+1} - t_i).$$

SPDE on $X_t^{s,x}$ –3

By virtue of standard estimates in stochastic analysis, it is possible to show that

$$\sup_i E|\tilde{z}_i - z_i|^2 \le \frac{C}{n^2}.$$

Hence, we get

$$X(X(x,t_i,t_{i+1}),t_{i+1},T) - X(x,t_{i+1},T) = X_x(x,t_{i+1},T)\tilde{z}_i + \int_0^1\int_0^1 \alpha X_{xx}(x+\alpha\beta z_i,t_{i+1},T)\tilde{z}_i^2 d\alpha d\beta + o(1/n),$$

where $o(1/n)$ is understood in the square mean sense. We have, $X_x(x,t_{i+1},T)\tilde{z}_i \approx X_x(x,t_{i+1},T)\sigma(x)(\Delta W_{t_i} + f(x)\Delta t_i)$; and $X_{xx}(x,t_{i+1},T)\tilde{z}_i^2 \approx X_{xx}(x,t_{i+1},T)\sigma^2(x)(\Delta W_{t_i})^2$.

SPDE on $X_t^{s,x}$ –3

So, after summation we obtain

$$\sum_i X_x(x, t_{i+1}, T)\sigma(x)\Delta W_{t_i} \overset{sq.mean}{\rightarrow} \int_t^T X_x(x, s, T)\sigma(x) \star dW_s,$$

$$\sum_i X_x(x, t_{i+1}, T)f(x)\Delta t_i \overset{sq.mean}{\rightarrow} \int_t^T X_x(x, s, T)f(x)ds,$$

and, finally,

$$\sum_i X_{xx}(x, t_{i+1}, T)a(x)(\Delta W_{t_i})^2 \overset{sq.mean}{\rightarrow} \int_t^T X_{xx}(x, s, T)a(x)ds.$$

In other words, we obtain (8), as required.

Ic: filtering SPDE

Reminder: diffusion filtering model

Consider an SDE system with smooth bounded coefficients

$$dX_t = f(X_t)dt + dW_t^1, \quad X_0 = x,$$
$$dY_t = h(X_t)dt + dW_t^2, \quad Y_0 = y, \tag{9}$$

where W^1, W^2 are independent Wiener processes. Here X is a *signal* and Y an observation process. Recall that the goal of the first part of this presentation concerns to the equation on the process

$$m_t := E[g(X_t)|\mathcal{F}_t^Y], \, 0 \le t \le T,$$

for some given function g and $T > 0$. For simplicity, both SDEs in this talk are in R^1, although any finite dimension may be allowed.

Bayes – Kallianpur – Striebel formula

We will use a well-known lemma, which is, in fact, a Bayes formula for conditional expectations.

Lemma

$$m_T \ (\equiv E[g(X_T)|\mathcal{F}^Y_T]) \ = \frac{\tilde{E}[g(X_T)\rho^{-1}|F^Y_T]}{\tilde{E}[\rho^{-1}|F^Y_T]},$$

where $\tilde{E}$ is the expectation with respect to measure $\tilde{P}$: $d\tilde{P} = \rho dP$, with

$$\rho \equiv \rho_{0T} = \exp(-\int_0^T h(X_t)dW_t^2 - (1/2)\int_0^T |h(X_t)|^2 dt).$$

Reminder: SPDE Theorem

Theorem

Let $g(x) \equiv x$. The process $m_T = E[g(X_T)|\mathcal{F}_T^Y]$ admits the following representation:

$$m_T = \frac{v^g(0,x)}{v^1(0,x)},$$

where $v^g(t,x)$ is a solution of the SPDE

$$-d_t v^g(t,x) = \left[\tfrac{1}{2} v^g_{xx}(t,x) + f(x) v^g_x(t,x)\right] dt$$

$$+h(x) v^g_x(t,x) \star dY_t, \quad 0 \le t \le T,$$

with initial data

$$v^g(T,x) = g(x), \quad x \in R^1.$$

Proof of Theorem (sketch) – 1

Reminders

Denote $v^h(s,x) = \tilde{E}[g(X_T^{s,x})\rho_{s,T}^{-1}|F_{s,T}^Y]$. Then,

$$v^h(T,x) = \tilde{E}[g(X_T^{T,x})\rho_{T,T}^{-1}|F_{T,T}^Y] = g(x).$$

Here

$$\rho \equiv \rho_{s,T} = \exp(-\int_s^T h(X_t)dw_t^2 - (1/2)\int_s^T |h(X_t)|^2 dt).$$

Recall that due to Girsanov's theorem the process $(Y_t,\, 0 \le t \le T)$ is a Wiener process on $(\Omega, F, (F_t,\, 0 \le t \le T), \tilde{P})$, in general, with nonzero starting value. Denote $\tilde{w}_t = Y_t$. Then on the space $(\Omega, F, (F_t,\, 0 \le t \le T), \tilde{P})$ our system (1) has the form

$$dX_t = f(X_t)dt + dw_t^1, \quad X_0 = x,$$
$$dY_t = d\tilde{w}_t, \quad Y_0 = y.$$

Proof of Theorem (sketch) – 2

In the sequel $v^g(s,x) = v(s,x)$. Actually, we would like to obtain for any $s < T$ the equality

$$v(s,x) - v(T,x) = \int_s^T h(x)v(t,x) \star d\tilde{w}_t + \int_s^T [(1/2)v_{xx}(t,x) + f(x)v_x(t,x)]dt, \qquad (10)$$

Let us use the identity

$$v(s,x) - v(T,x) = \sum_{i=1}^{N} (v(t_{i-1},x) - v(t_i,x))$$

for any partition $s = t_0 < t_1 < \ldots < t_N = T$. For one term from this sum we have,

$$v^h(t_{i-1},x) - v^h(t_i,x)$$
$$= \tilde{E}[\rho^{-1}_{t_{i-1},T} X(t_{i-1},T,x)|F^Y_{t_{i-1},T}] - \tilde{E}[\rho^{-1}_{t_i,T} X(t_i,T,x)|F^Y_{t_i,T}]$$
$$= \tilde{E}[\rho^{-1}_{t_{i-1},T} X(t_{i-1},T,x)|F^{\tilde{w}}_{t_{i-1},T}] - \tilde{E}[\rho^{-1}_{t_i,T} X(t_i,T,x)|F^{\tilde{w}}_{t_i,T}].$$

Proof of Theorem (sketch)

Now, due to evolution property of the family $X(s,T,x)$ (part Ib) we get (a.s.)

$$X(t_{i-1},T,x) = X(t_i,T,X_{t_i}^{t_{i-1},x})$$

$$= X_T^{t_{i-1},x} + X_x(t_i,T,x)(X_{t_i}^{t_{i-1},x} - x)$$

$$+(1/2)X_{xx}(t_i,T,x)(X_{t_i}^{t_{i-1}} - x)^2 + \alpha_i^1$$

$$= X_T^{t_i,x} + X_x(t_i,T,x)(f(x)\Delta t_i + \Delta w_{t_i}^1)$$

$$+(1/2)X_{xx}(t_i,T,x)\Delta t_i + \alpha_i^2,$$

where $\Delta t_i = t_i - t_{i-1}$, $\Delta w_{t_i}^j = w_{t_i}^j - w_{t_{i-1}}^j$, and $|\alpha_i^1| + |\alpha_i^2| = o(\Delta t_i)$ in a regular mean-square sense.

Thus,

$$\tilde{E}[\rho^{-1}_{t_{i-1},T} X(t_{i-1}, T, x) | F^{\tilde{w}}_{t_{i-1},T}]$$

$$= \tilde{E}[\rho^{-1}_{t_{i-1},T} X(t_i, T, X^{t_{i-1},x}_{t_i}) | \mathcal{F}^{\tilde{w}}_{t_{i-1},T}]$$

$$= \tilde{E}[\rho^{-1}_{t_{i-1},T} \{X^{t_i,x}_T + X_x(t_i, T, x) f(x) \Delta t_i$$

$$+ X_x((t_i, T, x) \Delta w^1_{t_i} + (1/2) X_{xx}(t_i, T, x) \Delta t_i\} | \mathcal{F}^{\tilde{w}}_{t_{i-1},T}] + \alpha^3_i.$$

Here again $\alpha^3_i = o(\Delta t_i)$.

Now, we would also like to replace here $\rho^{-1}_{t_{i-1},T}$ by $\rho^{-1}_{t_i,T}$, and eventually $\mathcal{F}^{\tilde{w}}_{t_{i-1},T}$ by $\mathcal{F}^{\tilde{w}}_{t_i,T}$.

Proof of Theorem (sketch, ctd)

Let us apply the same theorem 2 to the process $(X_t^{s,x},\ \rho_{s,t}^{-1},\ t \ge s)$. More precisely, let us note that this two-dimensional process satisfies the following SDE **system**:

$$
\begin{aligned}
dX_t^{s,x} &= f(X_t^{s,x})dt + dw_t^1, \quad X_s^{s,x} = x, \\
d\rho_{s,t}^{-1} &= h(X_t^{s,x})\rho_{s,t}^{-1} d\tilde{w}_t, \\
s \le t \le T&, \quad \rho_{s,s}^{-1} = 1.
\end{aligned} \tag{11}
$$

Consider the pair $Z = (X, \rho^{-1})$ as a random field satisfying an SD-system (11) & depending on the initial condition,

$$
Z_t^{s,x,\xi} = (X_t^{s,x,\xi}, (\rho^{-1})_t^{s,x,\xi}),
$$

where, in fact, $X_t^{s,x,\xi} = X_t^{s,x}$ and for brevity x will be dropped in $(\rho^{-1})_t^{s,x,\xi}$, although ρ^{-1} does depend on x via X.

Proof of Theorem (sketch)

Indeed, $\rho_{s,t}^{-1}$ has the following representation:

$$\rho_{s,t}^{-1} = \exp(\int_s^t h(X_r^{s,x}) d\tilde{w}_r - (1/2)\int_s^t |h(X_r^{s,x})|^2 dr).$$

Let us consider a bit more general set of processes $\{(X_t^{s,x}, \rho_{s,t}^{-1,\xi})\}$ which satisfy SDE's

$$dX_t^{s,x} = f(X_t^{s,x})dt + dw_t, \quad X_s^{s,x} = x,$$
$$d\rho_{s,t}^{-1,\xi} = h(X_t^{s,x})\rho_{s,t}^{-1,\xi} d\tilde{w}_t, \quad s \le t \le T, \quad \rho_{s,s}^{-1,\xi} = \xi, \tag{12}$$

with $X(t_i, T, x)i > 0$. Here $X_t^{s,x}$ is unchanged, and $\rho_t^{-1,X(t_i,T,x)i}$ reads,

$$\rho_{s,t}^{-1,\xi} = \xi \exp(\int_s^t h(X_r^{s,x}) d\tilde{w}_r - \frac{1}{2}\int_s^t |h(X_r^{s,x})|^2 dr) = \xi\rho_{s,t}^{-1}.$$

Proof of Theorem (sketch)

Due to theorem 2 we get,

$$-d_s\rho_{s,t}^{-1,\xi} = \left[\frac{1}{2}h^2(x)(\rho_{s,t}^{-1,\xi})^2(\rho_{s,t}^{-1,\xi})_{\xi\xi} + \frac{1}{2}(\rho_{s,t}^{-1,\xi})_{xx}\right.$$

$$\left.+f(x)(\rho_{s,t}^{-1,\xi})_x\right] ds + (\rho_{s,t}^{-1,\xi})_x \star dw_s^1 + h(x)(\rho_{s,t}^{-1,\xi})_\xi \star d\tilde{w}_s.$$

Since $(\rho_{s,t}^{-1,\xi})_\xi = \rho_{s,t}^{-1}$ and $(\rho_{s,t}^{-1,\xi})_{\xi\xi} = 0$, we have,

$$-d_s\rho_{s,t}^{-1,\xi} = \left[\frac{1}{2}(\rho_{s,t}^{-1,\xi})_{xx} + f(x)(\rho_{s,t}^{-1,\xi})_x\right] ds$$

$$+(\rho_{s,t}^{-1,\xi})_x \star dw_s^1 + h(x)\rho_{s,t}^{-1} \star d\tilde{w}_s.$$

So, we get with similar property for α_i^4 as for previous α_i,

$$\rho_{t_{i-1},T}^{-1,\xi} - \rho_{t_i,T}^{-1,\xi} = \left[\frac{1}{2}(\rho_{t_i,T}^{-1,\xi})_{xx} + f(x)(\rho_{t_i,T}^{-1,\xi})_x\right] \Delta t_i$$

$$+\rho_{t_i,T}^{-1,\xi}(-\Delta w_{t_i}^1) + h(x)\rho_{t_i,T}^{-1,\xi}(-\Delta\tilde{w}_{t_i}) + \alpha_i^4.$$

Proof of Theorem (sketch)

Below we use this assertion with $X(t_i, T, x)i = 1$. Now, we obtain

$$\tilde{E}[\rho^{-1}_{t_i,T} X(t_{i-1}, T, x) | \mathcal{F}^{\tilde{w}}_{t_{i-1},T}] = \tilde{E}[\{X^{t_i,x}_T$$

$$+(f(x)X_x(t_i, T, x) + \frac{1}{2}X_{xx}(t_i, T, x))\Delta t_i - X_x(t_i, T, x)\Delta w^1_{t_i}\}\times$$

$$\times\{\rho^{-1}_{t_i,T} + (\frac{1}{2}(\rho^{-1}_{t_i,T})_{xx} + f(x)(\rho^{-1}_{t_i,T})_x)\Delta t_i$$

$$+(\rho^{-1}_{t_i,T})_x(-\Delta w^1_{t_i}) + h(x)\rho^{-1}_{t_i,T}(-\Delta \tilde{w}_{t_i}) + \alpha^5_i\} | \mathcal{F}^{\tilde{w}}_{t_{i-1},T}].$$

Now, note that $\mathcal{F}^{\tilde{w}}_{t_{i-1},T} = \mathcal{F}^{\tilde{w}}_{t_{i-1},t_i} \bigvee \mathcal{F}^{\tilde{w}}_{t_i,T}$ and, moreover this σ-field is independent of w^1. So, using the regular calculus for conditional expectations, we get

$$\tilde{E}[X^{t_i,x}_T \rho^{-1}_{t_i,T} | \mathcal{F}^w_{t_{i-1},T}] = \tilde{E}[X^{t_i,x}_T \rho^{-1}_{t_i,T} | \mathcal{F}^w_{t_i,T}].$$

Proof of Theorem (sketch)

Similarly we replace $\mathcal{F}^w_{t_{i-1},T}$ by $\mathcal{F}^w_{t_i,T}$ in all all other terms. So, using $\tilde{E}(X_x(t_i,T,x)\Delta w^1_{t_i}|X_x(t_i,T,x)\Delta w^1_{t_i}) = 0$, etc., we get,

$$\tilde{E}[X_T^{t_{i-1},x}\rho^{-1}_{t_{i-1},T}|\mathcal{F}^{\tilde{w}}_{t_{i-1},T}] = \tilde{E}[X_T^{t_i,x}\rho^{-1}_{t_i,T}|\mathcal{F}^w_{t_{i-1},T}]$$

$$+\tilde{E}[\frac{1}{2}X_T^{t_i,x}(\rho^{-1}_{t_i,T})_{xx} + X_x(t_i,T,x)(\rho^{-1}_{t_i,T})_x + \frac{1}{2}X_{xx}\rho^{-1}_{t_i,T}|\mathcal{F}^w_{t_{i-1},T}]\Delta t_i$$

$$+\tilde{E}[f(x)(X(t_i,T,x)_x\rho^{-1}_{t_i,T} + X(t_i,T,x)(\rho^{-1}_{t_i,T})_x)|\mathcal{F}^w_{t_{i-1},T}]\Delta t_i$$

$$-\tilde{E}[X(t_i,T,x)h(x)\rho^{-1}_{t_i,T}\Delta\tilde{w}_{t_i}|\mathcal{F}^w_{t_{i-1},T}] + \alpha_i^6$$

$$\overset{+o(\Delta t_i)}{\approx} \tilde{E}[X(t_i,T,x)\rho^{-1}_{t_i,T}|\mathcal{F}^w_{t_i}] + \tilde{E}[\frac{1}{2}(X(t_i,T,x)\rho^{-1}_{t_i,T})_{xx}|\mathcal{F}^w_{t_i,T}]$$

$$+\tilde{E}[f(x)(X(t_i,T,x)\rho^{-1}_{t_i,T})_x|\mathcal{F}^w_{t_i,T}] - \Delta\tilde{w}_{t_i}\tilde{E}[h(x)X(t_i,T,x)\rho^{-1}_{t_i,T}|\mathcal{F}^w_{t_i,T}]$$

$$\overset{+o(\Delta t_i)}{\approx} v(t_i,x) + \frac{1}{2}v_{xx}(t_i,x)\Delta t_i + f(x)v_x(t_i,x)\Delta t_i - h(x)v(t_i,x)\Delta\tilde{w}_{t_i}.$$

Proof of Theorem (sketch)

We used, in particular, the possibility of changing order $\tilde{E}\partial_x = \partial_x \tilde{E}$.

After summation we have,

$$v(s,x) - v(T,x) = \sum_i h(x) v(t_i,x)(\tilde{w}_{t_{i-1}} - \tilde{w}_{t_i})$$

$$+ \sum_i \left\{ \frac{1}{2} v_{xx}(t_i,x) + f(x) v_x(t_i,x) \right\} \Delta_{t_i} + o(1).$$

Letting $\max \Delta t_i \to 0$, we get from here the integral equality

$$v(s,x) - v(T,x) = \int_s^T h(x) v(t,x) \star d\tilde{w}_t$$

$$+ \int_s^T [(1/2) v_{xx}(t,x) + f(x) v_x(t,x)] dt,$$

as required. The Theorem is proved.

This is the end of part I.